

# CLT for linear spectral statistics of random matrix $\mathbf{S}^{-1}\mathbf{T}$

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## Abstract

As a generalization of the univariate Fisher statistic, random Fisher matrices are widely-used in multivariate statistical analysis, e.g. for testing the equality of two multivariate population covariance matrices. The asymptotic distributions of several meaningful test statistics depend on the related Fisher matrices. Such Fisher matrices have the form  $\mathbf{F} = \mathbf{S}_y \mathbf{M} \mathbf{S}_x^{-1} \mathbf{M}^*$  where  $\mathbf{M}$  is a non-negative and non-random Hermitian matrix, and  $\mathbf{S}_x$  and  $\mathbf{S}_y$  are  $p \times p$  sample covariance matrices from two independent samples where the populations are assumed centred and normalized (i.e. mean 0, variance 1 and with independent components). In the large-dimensional context, Zheng (2012) establishes a central limit theorem for linear spectral statistics of a standard Fisher matrix where the two population covariance matrices are equal, i.e. the matrix  $\mathbf{M}$  is the identity matrix and  $\mathbf{F} = \mathbf{S}_y \mathbf{S}_x^{-1}$ . It is however of significant importance to obtain a CLT for general Fisher matrices  $\mathbf{F}$  with an arbitrary  $\mathbf{M}$  matrix. For the mentioned test of equality, null distributions of test statistics rely on a standard Fisher matrix with  $\mathbf{M} = I_p$  while under the alternative hypothesis, these distributions depends on a general Fisher matrix with arbitrary  $\mathbf{M}$ . As a first step to this goal, we propose in this paper a CLT for spectral statistics of the random matrix  $\mathbf{S}_x^{-1} \mathbf{T}$  for a general non-negative definite and **non-random** Hermitian matrix  $\mathbf{T}$  (note that  $\mathbf{T}$  plays the role of  $\mathbf{M}^* \mathbf{M}$ ). When  $\mathbf{T}$  is invertible, such a CLT can be directly derived using the CLT of Bai and Silverstein (2004) for the matrix  $\mathbf{T}^{-1} \mathbf{S}_x$ . However, in many large-dimensional statistic problems, the deterministic matrix  $\mathbf{T}$

is usually not invertible or has eigenvalues close to zero. The CLT from this paper covers this general situation.

# 1 Introduction

For a  $p \times p$  random matrix  $A_n$  with eigenvalues  $(\lambda_j)$ , linear spectral statistics (LSS) of type  $\frac{1}{p} \sum_j f(\lambda_j)$  for various test functions  $f$  are of central importance in the theory of random matrices and its applications. Central limit theorems (CLT) for such LSS of large dimensional random matrices have a long history, and received considerable attention in recent years. They have important applications in various domains like number theory, high-dimensional multivariate statistics and wireless communication networks; for more information, the readers are referred to the recent survey paper Johnstone (2007). To mention a few, in an early work, Jonsson (1982) gave a CLT for  $(\text{tr}(\mathbf{A}_n), \dots, \text{tr}(\mathbf{A}_n^k))$  for a sequence of Wishart matrices  $(\mathbf{A}_n)$ , where  $k$  is a fixed number, and the dimension  $p$  of the matrices grows proportionally to the sample size  $n$ . Subsequent works include Costin and Lebowitz (1995), Johansson (1998) which considered extensions of classical Gaussian ensembles, and Sinaï and Soshnikov (1998a,b) where Gaussian fluctuations are identified for LSS of Wigner matrices with a class of more general test functions. A general CLT for LSS of Wigner matrices was given in Bai and Yao (2005) where in particular, the limiting mean and covariance functions are identified. Similarly, Bai and Silverstein (2004) established a CLT for general sample covariance matrices with explicit limiting parameters. In Lytova and Pastur (2009), the authors reconsider such CLTs but with a new idea of interpolation that allows the generalisation from Gaussian matrix ensembles to matrix ensembles with general entries satisfying a moment condition. Recent improvements are proposed in Pan and Zhou (2008) that propose a generalization of the CLT in Bai and Silverstein (2004) (see also Wang and Yao (2013) for a complement on these CLT's). Finally, Pan (2012) and Bai and Zheng (2013) extend Bai and Silverstein (2004)'s CLT to biased and unbiased sample covariance matrices, respectively.

Random Fisher matrices are widely-used in multivariate statistical analysis, e.g. for testing the equality of two multivariate population covariance matrices. The asymptotic distributions of several meaningful test statistics depend on the related Fisher matrices. Such Fisher matrices have the form  $\mathbf{F} = \mathbf{S}_y \mathbf{M} \mathbf{S}_x^{-1} \mathbf{M}^*$  where  $\mathbf{M}$  is a non-negative deterministic Hermitian matrix, and  $\mathbf{S}_x$  and  $\mathbf{S}_y$  are  $p \times p$  sample covariance matrices from two independent samples where the populations are assumed centred and normalized (i.e. mean 0, variance 1 and with independent components). In the large-dimensional context,

Zheng (2012) establishes a CLT for linear spectral statistics of a standard Fisher matrix where the two population covariance matrices are equal, i.e. the matrix  $\mathbf{M}$  is the identity matrix and  $\mathbf{F} = \mathbf{S}_y \mathbf{S}_x^{-1}$ . It is however of significant importance to obtain a CLT for general Fisher matrices  $\mathbf{F}$  with an arbitrary  $\mathbf{M}$  matrix. For the mentioned test of equality, null distributions of test statistics rely on a standard Fisher matrix with  $\mathbf{M} = I_p$  while under the alternative hypothesis, these distributions depends on a general Fisher matrix with arbitrary  $\mathbf{M}$ .

In order to extend the CLT of Zheng (2012) to general Fisher matrices, we first need to establish limit theorems for the spectral (eigenvalues) distribution of the matrix  $\mathbf{M} \mathbf{S}_x^{-1} \mathbf{M}^*$ , or the matrix  $\mathbf{S}_x^{-1} \mathbf{T}$  where  $\mathbf{T} = \mathbf{M}^* \mathbf{M}$  is **non-random**. This includes i) an identification of the limit of its spectral distribution; ii) a CLT for its LSS. When the **non-random** matrix  $\mathbf{T}$  is invertible, since  $\mathbf{S}_x^{-1} \mathbf{T} = [\mathbf{T}^{-1} \mathbf{S}_x]^{-1}$ , CLT for LSS of  $\mathbf{S}_x^{-1} \mathbf{T}$  can be derived from the CLT of Bai and Silverstein (2004). However, in many large-dimensional statistic problems, the deterministic matrix  $\mathbf{T}$  is usually not invertible or has eigenvalues close to zero, and it is then hopeless to base the analysis on the CLT of Bai and Silverstein (2004).

In this paper, we consider the product  $\mathbf{S}_x^{-1} \mathbf{T}$  of a general determinist and **non-random** Hermitian matrix  $\mathbf{T}$  by the inverse  $\mathbf{S}_x^{-1}$  of a standard sample covariance matrix. As the main results of the paper, solutions to the aforementioned problems are provided.

The organization of this paper is as follows. Section 2 presents our main results. The proofs of these two main theorems are given in the following sections, respectively.

## 2 Main results

Following Bai and Silverstein (2004), let  $\{\mathbf{x}_t\}$ ,  $t = 1, \dots, n$  be a sequence of indepenent  $p$ -dimensional observations with independent and standardised components, i.e. for  $\mathbf{x}_t = (x_{tj})$ ,  $E x_{tj} = 0$  and  $E |x_{tj}|^2 = 1$ . The corresponding sample covariance matrix is

$$\mathbf{S} = \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t^* . \quad (2.1)$$

Consider the product matrix

$$\mathbf{S}^{-1} \mathbf{T} = \left( \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t^* \right)^{-1} \mathbf{T} , \quad (2.2)$$

where  $\mathbf{T}$  is a  $p \times p$  non-negative definite and non-random Hermitian matrix. Notice that we do not ask  $\mathbf{T}$  be invertible.

We first state the framework for our main results.

**Assumption 1** The  $p \times n$  observation matrix  $(x_{tj}, t = 1, \dots, n, j = 1, \dots, p)$  are made with independent elements satisfying  $E x_{tj} = 0$ ,  $E |x_{tj}|^2 = 1$ . Moreover, for any  $\eta > 0$  and as  $p, n \rightarrow \infty$ ,

$$\frac{1}{np} \sum_{t=1}^n \sum_{j=1}^p E [|x_{tj}|^2 I_{\{|x_{tj}| \geq \eta \sqrt{n}\}}] \rightarrow 0, \quad (2.3)$$

where  $I_{\{\cdot\}}$  is the indicator function.

The elements are either all real or all complex and we set an index  $\kappa = 1$  or  $\kappa = 2$ , respectively. In the later case,  $E\{x_{tj}^2\} = 0$  for all  $t, j$ .

**Assumption 1\*** In addition to Assumption 1, the entries  $\{x_{tj}\}$  have an uniform 4-th moment  $E|x_{tj}|^4 = 1 + \kappa$ . Moreover, for any  $\eta > 0$  and as  $p, n \rightarrow \infty$ ,

$$\frac{1}{np} \sum_{t=1}^n \sum_{j=1}^p E [|x_{tj}|^4 I_{\{|x_{tj}| \geq \eta \sqrt{n}\}}] \rightarrow 0. \quad (2.4)$$

**Assumption 1\*\*** In addition to Assumption 1, the entries  $\{x_{tj}\}$  have a finite 4-th moment (not necessarily the same). Moreover, for any  $\eta > 0$  and as  $p, n \rightarrow \infty$ ,

$$\frac{1}{np} \sum_{t=1}^n \sum_{j=1}^p E [|x_{tj}|^4 I_{\{|x_{tj}| \geq \eta \sqrt{n}\}}] \rightarrow 0. \quad (2.5)$$

**Assumption 2** The ESD  $H_n$  of  $\{\mathbf{T}\}$  tends to a limit  $H$ , which is a probability measure not degenerated to the Dirac mass at 0.

**Assumption 2\*** In addition to Assumption 2, the operator norm of  $\mathbf{T}$  is bounded when  $n, p \rightarrow \infty$ .

**Assumption 3** The dimension  $p$  and the sample size  $n$  both tend to infinity such that  $p/n \rightarrow y \in (0, 1)$ .

Assumption 1 states that the entries are independent, not necessarily identically distributed, but with homogeneous moments of first and second order, together with a Lindeberg type condition of order 2. Assumption 1\* reinforce Assumption 1 with similar conditions using a homogeneous fourth order moment that matches the Gaussian case. Assumption 1\*\* generalizes the previous one by allowing arbitrary values for the fourth moment of the entries.

Recall that the empirical spectral distribution (ESD) of a matrix is the distribution generated by its eigenvalues. When this ESD has a limit when the dimensions grow to infinity, the limit is called the limiting spectral distribution (LSD) of the matrix.

The first main result of the paper identifies the LSD of  $\mathbf{S}^{-1}\mathbf{T}$ .

**Theorem 2.1** *Under Assumptions 1, 2 and 3, with probability 1, the ESD  $F_n$  of  $\mathbf{S}^{-1}\mathbf{T}$  tends to a non-random distribution  $F^{y,H}$  whose Stieltjes transform  $s(z)$  is the unique solution to the equation*

$$zs(z) = -1 + \int \frac{tdH(t)}{-z - yz^2s(z) + t} . \quad (2.6)$$

*The distribution  $F^{y,H}$  is then the LSD of  $\mathbf{S}^{-1}\mathbf{T}$ .*

Next, we consider a LSS of  $\mathbf{S}^{-1}\mathbf{T}$  of form

$$F_n(f) = \int f(x)dF_n(x) = \frac{1}{p} \sum_{j=1}^p f(\lambda_j) ,$$

where the  $\{\lambda_j\}$ 's are the eigenvalues of the matrix  $\mathbf{S}^{-1}\mathbf{T}$  and  $f$  a given test function. Similarly to Bai and Silverstein (2004), a special feature here is that fluctuations of  $F_n(f)$  will not be considered around the LSD limit  $F^{y,H}(f)$ , but around  $F^{y_n, H_n}(f)$ , a finite-sample proxy of  $F^{y,H}$  obtained by substituting the parameters  $(y_n, H_n)$  to  $(y, H)$  in the LSD. Therefore, we consider the random variable

$$X_n(f) = p [F_n(f) - F^{y_n, H_n}(f)] = p \int f(x)d[F_n - F^{y_n, H_n}](x) .$$

The second main result of the paper is the following CLT.

**Theorem 2.2** *Assume that Assumptions 1\*, 2\* and 3 hold. Let  $f_1, \dots, f_k$  be functions analytic on an open domain of the complex plane enclosing the interval*

$$\left[ \frac{\liminf_p \lambda_{\min}^{\mathbf{T}}}{(1 + \sqrt{y})^2}, \frac{\limsup_p \lambda_{\max}^{\mathbf{T}}}{(1 - \sqrt{y})^2} \right] ,$$

*where  $\lambda_{\min}^{\mathbf{T}}$  and  $\lambda_{\max}^{\mathbf{T}}$  are respectively the smallest and the largest eigenvalue of  $\mathbf{T}$ . Then, the random vector  $[X_n(f_1), \dots, X_n(f_k)]$  weakly converges to a Gaussian vector  $[X_{f_1}, \dots, X_{f_k}]$  with mean function*

$$\mathbb{E}X_{f_j} = -\frac{\kappa - 1}{2\pi i} \oint f_j(z) \cdot \frac{1}{z^2} \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} dz , \quad (2.7)$$

*and covariance function*

$$\text{Cov}(X_{f_i}, X_{f_j}) = -\frac{\kappa}{4\pi^2} \oint \oint \frac{f_i(z_1)f_j(z_2) \cdot \frac{\partial[z_1(1+yz_1s(z_1))]}{\partial z_1} \frac{\partial[z_2(1+yz_2s(z_2))]}{\partial z_2}}{[z_1(1+yz_1s(z_1)) - z_2(1+yz_2s(z_2))]^2} dz_1 dz_2 \quad (2.8)$$

where  $\frac{1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z-1-yzs(z))^3}}{\left(1-y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z-1-yzs(z))^2}\right)^2} = \frac{1}{2} \frac{d \log \left(1-y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z-1-yzs(z))^2}\right)}{dz}$ . The contours in (2.7) and (2.8) are closed and are taken in the positive direction in the complex plane, all enclosing the support of  $F^{y,H}$ .

When the fourth moments of the entries are different from the value  $\kappa + 1$  matching the Gaussian case (3 or 2), the expression (4.5) has an additional term

$$\frac{1}{n^2} \sum_{i=1}^n b_i(z_1) b_i(z_2) \sum_{j=1}^p (E|X_{ij}|^4 - 1 - \kappa) \left[ E_{i-1} (z_1^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \left[ E_{i-1} (z_2^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj}$$

and the expression (4.14) has an additional term

$$-\frac{E(\beta_1(z))^2}{n^2 y} \sum_{j=1}^p (E|X_{1j}|^4 - 1 - \kappa) \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \right]_{jj} \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \mathbf{T} (z^{-1} \mathbf{T} - E\beta_1(z) \mathbf{I})^{-1} \right]_{jj}.$$

Then the covariance (4.9) and mean (4.16) will have additional terms, the limits of

$$\frac{\partial^2 \left\{ \frac{1}{n^2} \sum_{i=1}^n b_i(z_1) b_i(z_2) \sum_{j=1}^p (E|X_{ij}|^4 - 1 - \kappa) \left[ E_{i-1} (z_1^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \left[ E_{i-1} (z_2^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \right\}}{\partial z_1 \partial z_2}$$

and

$$\frac{y(1+yzs(z))^3}{z^2 \left(1-y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z-1-yzs(z))^2}\right)} \cdot \frac{1}{p} \sum_{j=1}^p \left\{ (E|X_{1j}|^4 - 1 - \kappa) \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \right]_{jj} \cdot \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \mathbf{T} (z^{-1} \mathbf{T} - E\beta_1(z) \mathbf{I})^{-1} \right]_{jj} \right\}.$$

When  $\frac{1}{p} \sum_{j=1}^p (E|X_{ij}|^4 - 1 - \kappa) \left[ (z_1^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \left[ (z_2^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj}$  converges to  $h(z_1, z_2)$  uniformly in  $i$ , then the covariance (4.9) will have the additional term

$$\frac{\partial^2 [y \cdot (1 + yz_1 s(z_1))(1 + yz_2 s(z_2))h(z_1, z_2)]}{\partial z_1 \partial z_2}$$

because  $E b_i(z) \rightarrow 1 + yzs(z)$  by (4.25).

Then Theorem 2.2 is easily extended to this situation as follows.

**Proposition 2.1** *Assume that Assumptions 1\*\*, 2\* and 3 hold. Let  $f_1, \dots, f_k$  be functions analytic on an open domain of the complex plane enclosing the interval*

$$\left[ \frac{\liminf_p \lambda_{\min}^{\mathbf{T}}}{(1 + \sqrt{y})^2}, \frac{\limsup_p \lambda_{\max}^{\mathbf{T}}}{(1 - \sqrt{y})^2} \right],$$

where  $\lambda_{\min}^{\mathbf{T}}$  and  $\lambda_{\max}^{\mathbf{T}}$  are respectively the smallest and the largest eigenvalue of  $\mathbf{T}$ . Moreover, assume in addition that the following non-random limits exist:

1.  $\frac{1}{p} \sum_{j=1}^p (E|X_{ij}|^4 - 1 - \kappa) \left[ (z_1^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \left[ (z_2^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj}$  converges to  $h(z_1, z_2)$  uniformly in  $i$ ;
2.  $\frac{1}{p} \sum_{j=1}^p (E|X_{1j}|^4 - 1 - \kappa) \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \right]_{jj} \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \mathbf{T} (z^{-1} \mathbf{T} - E\beta_1(z) \mathbf{I})^{-1} \right]_{jj}$  converges to  $h_M(z)$ .

Then the random vector  $[X_n(f_1), \dots, X_n(f_k)]$  weakly converges to a Gaussian vector  $[X_{f_1}, \dots, X_{f_k}]$  with mean function

$$\begin{aligned} EX_{f_j} &= -\frac{\kappa - 1}{2\pi i} \oint f_j(z) \cdot \frac{1}{z^2} \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} dz, \\ &\quad -\frac{1}{2\pi i} \int f_j(z) \cdot \frac{y(1+yzs(z))^3}{z^2 \left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)} \cdot h_M(z) dz \end{aligned}$$

and covariance function

$$\begin{aligned} Cov(X_{f_i}, X_{f_j}) &= -\frac{\kappa}{4\pi^2} \oint \oint \frac{f_i(z_1) f_j(z_2) \cdot \frac{\partial [z_1(1+yz_1s(z_1))]}{\partial z_1} \frac{\partial [z_2(1+yz_2s(z_2))]}{\partial z_2}}{[z_1(1+yz_1s(z_1)) - z_2(1+yz_2s(z_2))]^2} dz_1 dz_2 \\ &\quad -\frac{1}{4\pi^2} \oint \oint f_i(z_1) f_j(z_2) \cdot \frac{\partial^2 [y \cdot (1+yz_1s(z_1))(1+yz_2s(z_2))h(z_1, z_2)]}{\partial z_1 \partial z_2} dz_1 dz_2. \end{aligned}$$

The contours are closed and are taken in the positive direction in the complex plane, all enclosing the support of  $F^{y,H}$ .

When  $E|X_{ij}|^4 - 1 - \kappa = \beta_x + o(1)$  uniformly in  $i, j$  and  $\mathbf{T}$  is a diagonal matrix with positive eigenvalues, then we have

$$\begin{aligned} &\frac{1}{p} \sum_{j=1}^p (E|X_{1j}|^4 - 1 - \kappa) \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \right]_{jj} \left[ (z^{-1} \mathbf{T} - \mathbf{S}_1)^{-1} \mathbf{T} (z^{-1} \mathbf{T} - E\beta_1(z) \mathbf{I})^{-1} \right]_{jj} \\ \rightarrow &h_M(z) = \beta_x \cdot \int \frac{z^3 t dH(t)}{[t - z(1 + yzs(z))]^3} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{p} \sum_{j=1}^p (E|X_{ij}|^4 - 1 - \kappa) \left[ (z_1^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \left[ (z_2^{-1} \mathbf{T} - \mathbf{S}_i)^{-1} \right]_{jj} \\ &= h(z_1, z_2) = \beta_x \cdot \int \frac{z_1 z_2 dH(t)}{[t - z_1(1 + yz_1s(z_1))][t - z_2(1 + yz_2s(z_2))]} \end{aligned}$$

Then the mean (4.16) has the additional term

$$\frac{\beta_x \cdot yz(1 + yzs(z))^3}{\left(1 - y \int \frac{(1 + yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)} \cdot \int \frac{tdH(t)}{[t - z(1 + yzs(z))]^3}$$

and the covariance (4.9) has the additional term

$$\beta_x \cdot \frac{\partial^2}{\partial z_1 \partial z_2} \left[ y \int \frac{z_1(1 + yz_1s(z_1))z_2(1 + yz_2s(z_2))}{[t - z_1(1 + yz_1s(z_1))][t - z_2(1 + yz_2s(z_2))]} dH(t) \right].$$

Then Proposition 2.1 easily extends to the following proposition.

**Proposition 2.2** *Let assumptions of Proposition 2.1 hold. Moreover, assume that  $E|X_{ij}|^4 - 1 - \kappa = \beta_x + o(1)$  uniformly in  $i, j$  and  $\mathbf{T}$  is a diagonal matrix with positive eigenvalues, then we obtain that  $[X_n(f_1), \dots, X_n(f_k)]$  weakly converges to a Gaussian vector  $[X_{f_1}, \dots, X_{f_k}]$  with mean function*

$$\begin{aligned} EX_{f_j} &= -\frac{\kappa - 1}{2\pi i} \oint f_j(z) \cdot \frac{1}{z^2} \frac{y \int \frac{t(1 + yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1 + yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} dz, \\ &\quad -\frac{\beta_x}{2\pi i} \int \left[ f_j(z) \cdot \frac{yz(1 + yzs(z))^3}{1 - y \int \frac{(1 + yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}} \int \frac{tdH(t)}{[t - z(1 + yzs(z))]^3} \right] dz \end{aligned}$$

and covariance function

$$\begin{aligned} &Cov(X_{f_i}, X_{f_j}) \\ &= -\frac{\kappa}{4\pi^2} \oint \oint \frac{f_i(z_1)f_j(z_2) \cdot \frac{\partial[z_1(1 + yz_1s(z_1))]}{\partial z_1} \frac{\partial[z_2(1 + yz_2s(z_2))]}{\partial z_2}}{[z_1(1 + yz_1s(z_1)) - z_2(1 + yz_2s(z_2))]^2} dz_1 dz_2 \\ &\quad -\frac{\beta_x}{4\pi^2} \oint \oint \left\{ f_i(z_1)f_j(z_2) \cdot \frac{\partial^2}{\partial z_1 \partial z_2} \left[ y \int \frac{z_1(1 + yz_1s(z_1))z_2(1 + yz_2s(z_2))dH(t)}{[t - z_1(1 + yz_1s(z_1))][t - z_2(1 + yz_2s(z_2))]} \right] \right\} dz_1 dz_2. \end{aligned}$$

The contours in (2.7) and (2.8) are closed and are taken in the positive direction in the complex plane, all enclosing the support of  $F^{y,H}$ .

## 2.1 Relation between Theorem 2.2 and the CLT in Bai and Silverstein (2004)

Theorem 2.2 can be viewed a complement to the CLT in Bai and Silverstein (2004) while moving from the sample covariance matrix  $\mathbf{S}$  to its inverse  $\mathbf{S}^{-1}$ . When the factor  $\mathbf{T}$  in



$\mathbf{S}^{-1}\mathbf{T}$  is not invertible, these CLT's are not directly comparable. If  $\mathbf{T}$  is indeed invertible, these CLT's should be comparable. In this subsection, we will prove that they are indeed the same in this case. More precisely we prove that the mean and covariance functions given in Theorem 2.2 are the same as those given in Theorem 1.1 of Bai and Silverstein (2004).

Actually, when  $\mathbf{T}$  is invertible, we have

$$\begin{aligned}
s_n(z) &= \frac{1}{p} \text{tr}(\mathbf{S}^{-1}\mathbf{T} - z\mathbf{I})^{-1} = \frac{1}{p} \sum_{i=1}^p \frac{1}{\lambda_i((\mathbf{S}\mathbf{T}^{-1})^{-1}) - z} \\
&= \frac{1}{p} \sum_{i=1}^p \frac{1}{1/\lambda_i(\mathbf{S}\mathbf{T}^{-1}) - z} = \frac{1}{p} \sum_{i=1}^p \frac{\lambda_i(\mathbf{S}\mathbf{T}^{-1})}{1 - z\lambda_i(\mathbf{S}\mathbf{T}^{-1})} \\
&= \frac{-1}{pz} \sum_{i=1}^p \frac{\lambda_i(\mathbf{S}\mathbf{T}^{-1})}{\lambda_i(\mathbf{S}\mathbf{T}^{-1}) - \frac{1}{z}} \\
&= -\frac{1}{z} - \frac{1}{z^2} \cdot \frac{1}{p} \sum_{i=1}^p \frac{1}{\lambda_i(\mathbf{S}\mathbf{T}^{-1}) - \frac{1}{z}} \\
&= -\frac{1}{z} - \frac{1}{z^2} \cdot s_n^{\mathbf{S}\mathbf{T}^{-1}}\left(\frac{1}{z}\right) \\
&= -\frac{1}{z} - \frac{1}{z^2} \cdot \left( \frac{1}{y} \underline{m}_n\left(\frac{1}{z}\right) + \frac{1-y}{y} z \right) \\
&= -\frac{1}{yz} - \frac{1}{yz^2} \underline{m}_n\left(\frac{1}{z}\right),
\end{aligned}$$

where  $\underline{m}_n$  is the Stieltjes transform of  $\mathbf{X}_n \mathbf{T}^{-1} \mathbf{X}_n^*$  with  $\mathbf{X}_n^* = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  is  $p \times n$ . That is,

$$s(z) = -\frac{1}{yz} - \frac{1}{yz^2} \cdot \underline{m}\left(\frac{1}{z}\right), \quad 1 + yzs(z) = 1 - 1 - \frac{1}{z} \underline{m}\left(\frac{1}{z}\right) = -\frac{1}{z} \underline{m}\left(\frac{1}{z}\right), \quad (2.9)$$

where  $s(z)$  is the limit of  $s_n(z)$  and  $\underline{m}(z)$  is the limit of  $\underline{m}_n(z)$ . So the CLT of  $p(s_n(z) - s(z))$  is the same as  $-\frac{1}{z^2} \cdot n(\underline{m}_n(\frac{1}{z}) - \underline{m}(\frac{1}{z}))$ . By Lemma 1.1 of Bai and Silverstein (2004), we know that the CLT of

$$-\frac{1}{z^2} \cdot n(\underline{m}_n(\frac{1}{z}) - \underline{m}(\frac{1}{z})),$$

has mean

$$-\frac{1}{z^2} \frac{y \int \frac{t \cdot (\underline{m}(1/z))^3 d(H(t))}{(t + \underline{m}(1/z))^3}}{[1 - y \int \frac{(\underline{m}(1/z))^2 d(H(t))}{(t + \underline{m}(1/z))^2}]^2},$$

and covariance

$$\frac{1}{z_1^2 z_2^2} \frac{\underline{m}'(\frac{1}{z_1}) \underline{m}'(\frac{1}{z_2})}{(\underline{m}(\frac{1}{z_2}) - \underline{m}(\frac{1}{z_1}))^2} - \frac{1}{(z_1 - z_2)^2}.$$

It is easily to verify that

$$-\frac{1}{z^2} \frac{y \int \frac{t(\underline{m}(1/z))^3 d(H(t))}{(t+\underline{m}(1/z))^3}}{[1-y \int \frac{(\underline{m}(1/z))^2 d(H(t))}{(t+\underline{m}(1/z))^2}]^2} = \frac{1}{z^2} \frac{y \int \frac{t \cdot z^3(1+ys(z))^3 dH(t)}{(t-z(1+yzs(z)))^3}}{[1-y \int \frac{z^2(1+yzs(z))^2 dH(t)}{(t-z(1+yzs(z)))^2}]^2} = \frac{1}{z^2} \frac{y \int \frac{t \cdot (1+ys(z))^3 dH(t)}{(t/z-1-yzs(z))^3}}{[1-y \int \frac{(1+ys(z))^2 dH(t)}{(t/z-1-yzs(z))^2}]^2},$$

and

$$\frac{1}{z_1^2 z_2^2} \frac{\underline{m}'(\frac{1}{z_1}) \underline{m}'(\frac{1}{z_2})}{(\underline{m}(\frac{1}{z_2}) - \underline{m}(\frac{1}{z_1}))^2} - \frac{1}{(z_1 - z_2)^2} = \frac{[z_1(1 + yz_1s(z_1))]'[z_2(1 + yz_2s(z_2))]' }{[z_1(1 + yz_1s(z_1)) - z_2(1 + yz_2s(z_2))]^2} - \frac{1}{(z_1 - z_2)^2},$$

which are the same as given in Theorem 2.2. Thus, when  $\mathbf{T}$  is inversible, the CLT of LSS of  $\mathbf{S}^{-1}\mathbf{T}$  has the same mean and covariance functions as that obtained by Theorem 1.1 of Bai and Silverstein (2004).

### 3 Proof of Theorem 2.1

Using exactly the same approach employed in Section 4.3 of Bai and Silverstein (2010), we may truncate the extreme eigenvalues of  $\mathbf{T}$  and tails of the random variables  $x_{ij}$  and then renormalize them without altering the LSD of  $\mathbf{S}^{-1}\mathbf{T}$ . So we may assume that Assumption 2\* is true and  $|x_{ij}| \leq \eta_n \sqrt{n}$  where  $\eta_n \rightarrow 0$ .

Now, we proceed with the proof of Theorem 2.1. To start with, we assume that  $\mathbf{T}$  is invertible and there is a positive constant  $\omega > 0$  such that  $H(\omega) = 0$ , that is, the norm of  $\mathbf{T}^{-1}$  is bounded. By Theorem 4.1 of Bai and Silverstein (2010) we know that the LSD of  $\mathbf{S}\mathbf{T}^{-1}$  exists and its Stieltjes transform  $m(z)$  satisfies

$$m(z) = \int \frac{1}{t(1-y-yzm(z))-z} dH(1/t) = \int \frac{tdH(t)}{1-y-yzm(z)-tz}. \quad (3.1)$$

Note that  $m(z)$  is the unique solution to the equation (3.1) that has the same sign of imaginary part as  $z$ .

If we denote the Stieltjes transforms of  $\mathbf{S}^{-1}\mathbf{T}$  and  $\mathbf{S}\mathbf{T}^{-1}$  by  $s_n(z)$  and  $m_n(z)$ , respectively. By the relation

$$m_n(z) = -\frac{1}{z} - \frac{1}{z^2} s_n(1/z),$$

and  $m_n(z) \rightarrow m(z)$  a.s., we know that with probability 1,  $s_n(z)$  converges to a limit  $s(z)$  that satisfies

$$-\frac{1}{z} - \frac{1}{z^2} s(1/z) = \int \frac{tdH(t)}{1-y-yz(-\frac{1}{z} - \frac{1}{z^2} s(1/z)) - tz}. \quad (3.2)$$

Changing  $z$  as  $1/z$  and simplifying it, we obtain (2.6).

Now, we consider possibly singular  $\mathbf{T}$  and will show that for any fixed  $z = u + iv$  with  $v > 0$ ,  $s_n(z)$  still converges to a limit  $s(z)$  that satisfies (2.6).

For any fixed  $\varepsilon > 0$ , define  $\mathbf{T}_\varepsilon = \mathbf{T} + \varepsilon \mathbf{I}$  and define  $\mathbf{S}_+$  from  $\mathbf{S}$  by replacing its eigenvalues less than  $\frac{1}{2}a$  as  $\frac{1}{2}a$ , where  $a = (1 - \sqrt{y})^2$ . By the rank inequality of Bai (1999), we have

$$\left\| F^{\mathbf{S}^{-1}\mathbf{T}} - F^{\mathbf{S}_+^{-1}\mathbf{T}} \right\| \leq \frac{1}{p} \# \left\{ \lambda_i(\mathbf{S}) \leq \frac{1}{2}a \right\} \rightarrow 0, a.s. \quad (3.3)$$

By Theorem A.45 of Bai and Silverstein (2010),

$$L(F^{\mathbf{S}_+^{-1}\mathbf{T}}, F^{\mathbf{S}_+^{-1}\mathbf{T}_\varepsilon}) \leq \|\mathbf{S}_+^{-1}(\mathbf{T} - \mathbf{T}_\varepsilon)\| \leq 2a^{-1}\varepsilon. \quad (3.4)$$

Using again the rank inequality, we have

$$\left\| F^{\mathbf{S}^{-1}\mathbf{T}_\varepsilon} - F^{\mathbf{S}_+^{-1}\mathbf{T}_\varepsilon} \right\| \leq \frac{1}{p} \# \left\{ \lambda_i(\mathbf{S}) \leq \frac{1}{2}a \right\} \rightarrow 0, a.s. \quad (3.5)$$

By what has been proved anove for invertible  $\mathbf{T}$ , with probability 1,  $s_{n,\varepsilon}(z) = \frac{1}{p}\text{tr}(\mathbf{S}^{-1}\mathbf{T}_\varepsilon) \rightarrow s_\varepsilon(z)$  which is a solution to the equation

$$zs_\varepsilon(z) = -1 + \int \frac{tdH_\varepsilon(t)}{1 - z - yz^2s_\varepsilon(z) + t}. \quad (3.6)$$

where  $H_\varepsilon(t) = H(t - \varepsilon)$ .

To complete the proof of the theorem, we only need to verify that the equation (3.6) has a unique solution that is the Stieltjes transform of a probability distribution, and the solution  $s_\varepsilon(z)$  is right-continuous at  $\varepsilon = 0$ . Making a transformation  $w_\varepsilon(z) = \sqrt{z}(1 + zs_\varepsilon(z))$ , where  $\sqrt{z}$  is the square root of  $z$  satisfying  $\Im(z)\Im(\sqrt{z}) > 0$ , then the equation (3.6) becomes

$$w_\varepsilon(z) = \int \frac{tdH_\varepsilon(t)}{\frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z)}, \quad (3.7)$$

where  $w_\varepsilon(z)$  has the same sign of imaginary part as  $z$ .

We only need to consider the case where  $\Im(z) > 0$ . Let  $w_2 = \Im(w_\varepsilon(z)) > 0$ , comparing the imaginary parts of (3.7), we have

$$\begin{aligned} w_2 &= \int \frac{\frac{(1+t)\Im(\sqrt{z})}{|z|} + (1-y)\Im(\sqrt{x}) + yw_2}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right|^2} dH_\varepsilon(t) \\ &> \int \frac{yw_2}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right|^2} dH_\varepsilon(t), \end{aligned}$$

which implies that

$$\int \frac{y dH_\varepsilon(t)}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right|^2} < 1. \quad (3.8)$$

Suppose (3.7) had two solution  $w^{(j)}$  with  $w_2^{(j)} = \Im(w^{(j)}) > 0$ ,  $j = 1, 2$ . Then making difference of both sides and cancelling  $w_1 - w_2$  from both sides, we obtain

$$1 = y \int \frac{tdH_\varepsilon(t)}{\left( \frac{1+t}{\sqrt{z} - (1-y)\sqrt{z} - yw^{(1)}} \right) \left( \frac{1+t}{\sqrt{z} - (1-y)\sqrt{z} - yw^{(2)}} \right)},$$

which implies by Cauchy-Schwarz that

$$1 \leq \left( \int \frac{y dH_\varepsilon(t)}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw^{(1)} \right|^2} \int \frac{y dH_\varepsilon(t)}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw^{(2)} \right|^2} \right)^{1/2} < 1,$$

where the last inequality follows by applying (3.8) for both  $w^{(1)}$  and  $w^{(2)}$ . The contradiction proves the uniqueness of a solution to (3.7).

Finally, we show that the solution  $w_\varepsilon$  is right-continuous at  $\varepsilon = 0$ . By (3.7), we have

$$w_\varepsilon(z) - w_0(z) = \frac{\int \frac{td(H_\varepsilon(t) - H(t))}{\frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z)}}{1 - y \int \frac{tdH(t)}{\left( \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right) \left( \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_0(z) \right)}}. \quad (3.9)$$

Since

$$\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right| \geq -\Im\left(\frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z)\right) > (1-y)\Im(\sqrt{z}),$$

we have

$$\begin{aligned} & \left| y \int \frac{tdH(t)}{\left( \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_\varepsilon(z) \right) \left( \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_0(z) \right)} \right| \\ & \leq \left( y \int \frac{tdH(t)}{\left| \frac{1+t}{\sqrt{z}} - (1-y)\sqrt{z} - yw_0(z) \right|^2} \right)^{1/2} < 1. \end{aligned}$$

It follows that  $w_\varepsilon(z) - w_0(z) \rightarrow 0$  which implies that  $s_\varepsilon(z) - s(z) \rightarrow 0$ .

The proof of Theorem 2.1 is complete.

## 4 Proof of Theorem 2.2

We first describe the strategy of the proof that follows the proof in Bai and Silverstein (2004) and an improved version in Bai and Silverstein (2010). First, due to Assumption 1\*, we may truncate the random variables  $x_{ij}$  at  $\eta_n\sqrt{n}$  and renormalize them without alerting the CLT of  $X_n(f)$ , where  $\eta_n \downarrow 0$  with some slow rate. Therefore, we may make the following additional assumptions:

1.  $|x_{ij}| \leq \eta_n\sqrt{n}$ ;
2.  $\mathbb{E}x_{ij}^2 = \kappa - 1 + o(n^{-1})$ ;
3.  $\mathbb{E}|x_{ij}^4| = 1 + \kappa + o(1)$ .

Define a contour  $\mathcal{C}_n$  by

$$\mathcal{C}_n = \mathcal{C}_l \cup \mathcal{C}_u \cup \mathcal{C}_b \cup \mathcal{C}_r$$

where

$$\begin{aligned} \mathcal{C}_u &= \{x + i\nu_0 : x \in [x_l, x_u]\}, & \mathcal{C}_b &= \{x - i\nu_0 : x \in [x_l, x_u]\}, \\ \mathcal{C}_l &= \{x_l + i\nu : |\nu| \leq \nu_0\}, & \mathcal{C}_r &= \{x_r + i\nu : |\nu| \leq \nu_0\} \end{aligned}$$

and  $(x_l, x_r) \supset [\liminf \lambda_{\min}(\mathbf{T})/(1 + \sqrt{y})^2, \limsup \lambda_{\max}(\mathbf{T})/(1 - \sqrt{y})^2]$  and is enclosed in the analytic region of the  $f_j(x)$ 's. Following Bai and Silverstein (2004), we can rewrite  $X_n(f)$  as

$$X_b(f) = -\frac{1}{2\pi} \oint_{\mathcal{C}_n} f(z) p(s_n(z) - s_n^0(z)) dz, \quad (4.1)$$

where  $s_n^0(z)$  is the Stieltjes transform of  $F^{y_n, H_n}$ .

**Remark 4.1** *Note that the identity (4.1) holds only when all eigenvalues of  $\mathbf{S}^{-1}\mathbf{T}$  are falling inside the interval  $(x_l, x_r)$ . By Bai and Yin (1993), with probability 1, when  $n$  is large, all eigenvalues of  $\mathbf{S}$  are falling inside the interval  $((1 - \sqrt{y})^2 - \varepsilon, (1 + \sqrt{y})^2 + \varepsilon)$  which confirms that (4.1) holds for all large  $n$ . So, without loss of generality, in the proof of Theorem 2.2, we assume (4.1) holds.*

Write  $M_n(z) = p(s_n(z) - s_n^0(z)) = M_n^1(z) + M_n^2(z)$ , where  $M_n^1(z) = p(s_n(z) - \mathbb{E}s_n(z))$  and  $M_n^2(z) = p(\mathbb{E}s_n(z) - s_n^0(z))$ . We shall establish a CLT for  $M_n^1(z)$ , and then find the limit of  $M_n^2(z)$  on  $\mathcal{C}_u$  and  $\mathcal{C}_b$ . Their combinaison will complete the proof of Theorem 2.2.

## 4.1 Finite-dimensional convergence of $M_n^1(z)$ on $\mathcal{C}_u$

We first prove an auxiliary theorem.

**Theorem 4.1** *Under Assumptions 1\*, 2\*, and 3,  $M_n^1(z)$  converges weakly to a complex Gaussian process  $M_1(\cdot)$  on the contour  $z \in \mathcal{C}$ , with mean function*

$$\mathbb{E}M_1(z) = 0$$

*and covariance function*

$$\text{Cov}(M_1(z_1), M_1(z_2)) = \kappa \left[ \frac{[z_1(1 + yz_1s(z_1))]'[z_2(1 + yz_2s(z_2))]' }{[z_1(1 + yz_1s(z_1)) - z_2(1 + yz_2s(z_2))]^2} - \frac{1}{(z_1 - z_2)^2} \right]. \quad (4.2)$$

PROOF. Let  $\mathbb{E}_i$  denote the conditional expectation given  $\{\mathbf{x}_1, \dots, \mathbf{x}_i\}$  and  $\mathbb{E}_0$  denote the unconditional expectation. Denote  $z = u + iv$  with  $v > 0$  fixed,

$$\begin{aligned} \boldsymbol{\alpha}_i &= \frac{1}{\sqrt{n}}\mathbf{X}_i, \quad \mathbf{S}_k = \mathbf{S} - \boldsymbol{\alpha}_k\boldsymbol{\alpha}_k^*, & \mathbf{D} &= \mathbf{S} - z\mathbf{I}, \quad \mathbf{D}_k = \mathbf{T} - z\mathbf{S}_k, \\ \beta_i(z) &= \frac{1}{1 - \boldsymbol{\alpha}_i'(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}\boldsymbol{\alpha}_i}, & \bar{\beta}_i(z) &= \frac{1}{1 - \frac{1}{n}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}}, \\ \bar{\beta}_i(z, \theta_i) &= \frac{1}{1 - \frac{1}{n}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1} + \theta_i\hat{\gamma}_i(z)}, & \hat{\gamma}_i(z) &= \frac{1}{n}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1} - \boldsymbol{\alpha}_i'(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}\boldsymbol{\alpha}_i. \end{aligned}$$

Then we have

$$\beta_i(z)^{-1} = \bar{\beta}_i(z)^{-1} - \hat{\gamma}_i(z), \quad \bar{\beta}_i(z)\beta_i^{-1}(z) = 1 - \bar{\beta}_i(z)\hat{\gamma}_i(z).$$

Therefore, by Taylor expansion

$$\begin{aligned} & (\mathbb{E}_i - \mathbb{E}_{i-1}) \log \beta_i^{-1}(z) \\ &= (\mathbb{E}_i - \mathbb{E}_{i-1}) (\log \beta_i^{-1}(z) - \log \bar{\beta}_i^{-1}(z)) \\ &= (\mathbb{E}_i - \mathbb{E}_{i-1}) [-\bar{\beta}_i(z)\hat{\gamma}_i(z) + \bar{\beta}_i^2(z, \theta_i)\hat{\gamma}_i^2(z)] \\ &= -\mathbb{E}_i\bar{\beta}_i(z)\hat{\gamma}_i(z) + (\mathbb{E}_i - \mathbb{E}_{i-1})\bar{\beta}_i^2(z, \theta_i)\hat{\gamma}_i^2(z). \end{aligned} \quad (4.3)$$

Here, we have used a formula that  $\log \beta_i^{-1} - \log \bar{\beta}_i^{-1} = \log \bar{\beta}_i(z)\beta_i^{-1}$ . In fact we should add an additional term  $2\pi k(z)$  where  $k(z)$  is a random integer function of  $z$ . This term does make any contribution because we only need the derivative of the function  $\log \beta_i^{-1}$  in the next step.

For any  $i \leq n$ , we have

$$\begin{aligned}
s_n(z) &= \frac{1}{p} \text{tr}(\mathbf{S}^{-1} \mathbf{T} - z \mathbf{I})^{-1} = \frac{1}{p} \text{tr} \mathbf{S} (\mathbf{T} - z \mathbf{S})^{-1} \\
&= \frac{1}{p} \text{tr}(\mathbf{S}_i + \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i') (\mathbf{T} - z \mathbf{S}_i - z \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i')^{-1} \\
&= \frac{1}{pz} \text{tr}(z \mathbf{S}_i + z \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i') \left( (\mathbf{T} - z \mathbf{S}_i)^{-1} + \frac{z(\mathbf{T} - z \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' (\mathbf{T} - z \mathbf{S}_i)^{-1}}{1 - z \boldsymbol{\alpha}_i' (\mathbf{T} - z \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i} \right) \\
&= -\frac{1}{pz} \text{tr}(\mathbf{T} - z \mathbf{S}_i - z \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' - \mathbf{T}) \left( (\mathbf{T} - z \mathbf{S}_i)^{-1} + \frac{z(\mathbf{T} - z \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' (\mathbf{T} - z \mathbf{S}_i)^{-1}}{1 - z \boldsymbol{\alpha}_i' (\mathbf{T} - z \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i} \right) \\
&= -\frac{1}{pz} \left( p + \frac{z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i}{1 - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} - \frac{z^2 (\boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i)^2}{1 - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \right) \\
&\quad + \frac{1}{pz} \text{tr} \left( \mathbf{T} \mathbf{D}_i^{-1} + \frac{z \mathbf{T} \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \right) \\
&= -\frac{1}{z} + \frac{1}{pz} \text{tr} \mathbf{T} \mathbf{D}_i^{-1} + \frac{1}{p} \frac{\boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \mathbf{T} \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i}{1 - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \\
&= -\frac{1}{z} + \frac{1}{pz} \text{tr} \mathbf{T} \mathbf{D}_i^{-1} + \frac{1}{p} \frac{\partial}{\partial z} \log \beta_i(z) .
\end{aligned}$$

Therefore,

$$\begin{aligned}
s_n(z) - \mathbb{E} s_n(z) &= -\frac{1}{p} \frac{\partial}{\partial z} \sum_{i=1}^n (\mathbb{E}_i - \mathbb{E}_{i-1}) \log(1 - z \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i) \\
&= -\frac{1}{p} \frac{\partial}{\partial z} \sum_{i=1}^n \mathbb{E}_i \bar{\beta}_i(z) \hat{\gamma}_i(z) - \frac{1}{p} \frac{\partial}{\partial z} \sum_{i=1}^n (\mathbb{E}_i - \mathbb{E}_{i-1}) \bar{\beta}_i^2(z, \theta_i) \hat{\gamma}_i^2(z) .
\end{aligned}$$

Since

$$\begin{aligned}
&E \left| \sum_{i=1}^n (\mathbb{E}_{i-1} - \mathbb{E}_i) \bar{\beta}_i^2(z, \theta_i) \hat{\gamma}_i^2(z) \right|^2 \\
&= \sum_{i=1}^n E \left| (\mathbb{E}_{i-1} - \mathbb{E}_i) \bar{\beta}_i^2(z, \theta_i) \hat{\gamma}_i^2(z) \right|^2 \\
&\leq 4 \sum_{i=1}^n E \left| \bar{\beta}_i^2(z, \theta_i) \hat{\gamma}_i^2(z) \right|^2 \\
&\leq 4 \sum_{i=1}^n E |\hat{\gamma}_i(z)|^4 \\
&= n O(n^{-1} \eta_n^4) = o(1) \quad (\text{by Lemma 4.1})
\end{aligned} \tag{4.4}$$

we have

$$p((s_n(z) - \mathbb{E} s_n(z))) = - \sum_{i=1}^n \mathbb{E}_i \frac{d}{dz} \bar{\beta}_i(z) \hat{\gamma}_i(z) + o_p(1) = - \frac{d}{dz} \sum_{i=1}^n Y_i(z) + o_p(1) ,$$

where

$$Y_i(z) = \mathbb{E}_i \bar{\beta}_i(z) \hat{\gamma}_i(z) .$$

We first consider a finite sum

$$\sum_{k=1}^r \sum_{i=1}^n a_k Y_i(z_k) = \sum_{i=1}^n \sum_{k=1}^r a_k Y_i(z_k)$$

from  $r$  points  $z_k$  on the contour with arbitrary weighting numbers  $a_k$ . That is, we need to complete the following two steps:

*Step 1:* Verify the Lyapunoff condition, i.e.  $\sum_{i=1}^n \mathbb{E} |Y_i(z)|^4 = o(1)$ .

In fact, if  $z \in \mathcal{C}_u$  or  $\mathcal{C}_b$ , by the fact that  $|\bar{\beta}_i(z)| < |z|/\nu_0$ ,

$$\begin{aligned} \sum_{i=1}^n \mathbb{E} |Y_i(z)|^4 &\leq C \sum_{i=1}^n \mathbb{E} |\hat{\gamma}_i(z)|^4 \\ &\leq C n^{-4} \sum_{i=1}^n \mathbb{E} \left[ \left( \text{tr}(\mathbf{D}_i^{-1}(z) \mathbf{D}_i^{-1}(\bar{z})) \right)^2 + \max_{ij} \mathbb{E} |x_{ij}^8| \sum_{j=1}^p |[\mathbf{D}_i^{-1}]_{jj}|^4 \right] \\ &\leq C(n^{-1} + \eta_n^4) \rightarrow 0, \end{aligned}$$

where  $[\mathbf{D}_i^{-1}]_{jj}$  is the  $j$ -th diagonal entry of  $\mathbf{D}_i^{-1}(z)$  which is bounded by  $|z|/\nu_0$ .

*Step 2:* Find the limits of  $\frac{\partial^2}{\partial z_1 \partial z_2} \sum_{i=1}^n \mathbb{E}_{i-1} Y_i(z_1) Y_i(z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \sum_{i=1}^n \mathbb{E}_{i-1} [\mathbb{E}_i \bar{\beta}_i(z_1) \hat{\gamma}_i(z_1) \cdot \mathbb{E}_i \bar{\beta}_i(z_2) \hat{\gamma}_i(z_2)]$ .

$$E \left| \sum_{i=1}^n \sum_{k=1}^r a_k Y_i(z_k) \right|^2 = \sum_{i=1}^n E \left| \sum_{k=1}^r a_k Y_i(z_k) \right|^2 \leq K \sum_{i=1}^n \sum_{k=1}^r |a_k|^2 E |Y_i(z_k)|^2 \leq K$$

because  $E |Y_i(z_k)|^2 = O(n^{-1})$  by Lemma 4.1. We have

$$\bar{\beta}_i(z) - b_i(z) = \bar{\beta}_i(z) b_i(z) \left( n^{-1} \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1} - n^{-1} E \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1} \right)$$

where  $b_i(z) = \frac{1}{1 - n^{-1} E \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1}}$ . Then (Bai and Silverstein (2010), P139)

$$\begin{aligned} E |\bar{\beta}_i(z) - b_i(z)|^{2l} &\leq K E \left( n^{-1} \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1} - n^{-1} E \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1} \right)^{2l} \\ &= K E \left( \frac{1}{n} \sum_{j=2}^n [\mathbb{E}_j \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_1)^{-1} - \mathbb{E}_{j-1} \text{tr}(\frac{1}{z} \mathbf{T} - \mathbf{S}_1)^{-1}] \right)^{2l} \\ &= K E \left( \frac{1}{n} \sum_{j=2}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{tr} \left\{ \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1j} \right)^{-1} \right\} \right)^{2l} \\ &= K E \left( \frac{1}{n} \sum_{j=2}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \frac{\alpha'_j (\frac{1}{z} \mathbf{T} - \mathbf{S}_{1j})^{-2} \alpha_j}{1 + \alpha'_j (\frac{1}{z} \mathbf{T} - \mathbf{S}_{1j})^{-1} \alpha_j} \right)^{2l} \\ &\leq \frac{K}{n^{2l}} E \left( \sum_{j=2}^n \left| (\mathbb{E}_j - \mathbb{E}_{j-1}) \frac{\alpha'_j (\frac{1}{z} \mathbf{T} - \mathbf{S}_{1j})^{-2} \alpha_j}{1 + \alpha'_j (\frac{1}{z} \mathbf{T} - \mathbf{S}_{1j})^{-1} \alpha_j} \right|^2 \right)^l \\ &\quad \text{(Lemma 2.12 of Bai and Silverstein (2010))} \\ &\leq \frac{K}{n^l \cdot \nu^{2l}}. \end{aligned}$$

Then  $E |\bar{\beta}_i(z) - b_i(z)|^{2l} = O(n^{-l})$  is uniformly. Then

$$\sum_{i=1}^n \mathbb{E}_{i-1} [\mathbb{E}_i \bar{\beta}_i(z_1) \hat{\gamma}_i(z_1) \cdot \mathbb{E}_i \bar{\beta}_i(z_2) \hat{\gamma}_i(z_2)] - \sum_{i=1}^n b_i(z_1) b_i(z_2) \mathbb{E}_{i-1} [\mathbb{E}_i \hat{\gamma}_i(z_1) \cdot \mathbb{E}_i \hat{\gamma}_i(z_2)] = o_p(1)$$



Then we only consider the limit of

$$\begin{aligned}
& \sum_{i=1}^n b_i(z_1) b_i(z_2) \mathbf{E}_{i-1} [\mathbf{E}_i \hat{\gamma}_i(z_1) \mathbf{E}_i \hat{\gamma}_i(z_2)] \\
&= \frac{\kappa}{n^2} \sum_{i=1}^n b_i(z_1) b_i(z_2) \text{tr} \left[ \mathbf{E}_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_i \right)^{-1} \mathbf{E}_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} \right] \\
& \quad [\text{by (1.15) of Bai and Silverstein (2004)}].
\end{aligned} \tag{4.5}$$

We have

$$\left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right) - \frac{1}{z} \mathbf{T} + \frac{n-1}{n} b_i(z) \mathbf{I} = - \sum_{k \neq i} \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* + \frac{n-1}{n} b_i(z) \mathbf{I}.$$

Multiplying by  $(\frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T})^{-1}$  on the left,  $(\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1}$  on the right, then we have

$$\begin{aligned}
\left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} &= - \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \\
&\quad - \sum_{k \neq i} \beta_{k(i)}(z) \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \\
&\quad + \frac{n-1}{n} b_i(z) \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \\
&= - \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} - b_i(z) \mathbf{A}(z) - \mathbf{B}(z) - \mathbf{C}(z)
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\mathbf{A}(z) &= \sum_{k \neq i} \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} (\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I}) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \\
\mathbf{B}(z) &= \sum_{k \neq i} (\beta_{k(i)}(z) - b_i(z)) \left( \frac{n-1}{n} b_j(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \\
\mathbf{C}(z) &= \frac{1}{n} b_i(z) \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \sum_{k \neq i} \left[ \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \right]
\end{aligned}$$

Similarly, we have

$$\left\| \left( \frac{n-1}{n} b_j(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \right\| \leq K$$

where  $K$  is a constant. Similarly, we have

$$\begin{aligned}
|b_{ik}(z) - b_i(z)| &= \left| b_i(z) b_{ik}(z) \left[ \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} - \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right] \right| \\
&= \frac{1}{n} \left| b_{ik}(z) b_i(z) \mathbf{E} \beta_{k(i)}(z) \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-2} \boldsymbol{\alpha}_k \right| = O(n^{-1})
\end{aligned} \tag{4.7}$$

and

$$\mathbb{E}(\beta_{k(i)}(z) - b_{ik}(z))^2 = O(n^{-1}) \text{ by Lemma 4.1} \quad (4.8)$$

where

$$\beta_{k(i)}(z) = \frac{1}{1 - \boldsymbol{\alpha}'_k(\frac{1}{z}\mathbf{T} - \mathbf{S}_{ik})^{-1}\boldsymbol{\alpha}_k}$$

$$b_i(z) = \frac{1}{1 - \frac{1}{n}\mathbb{E}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}}, \quad b_{ik}(z) = \frac{1}{1 - \frac{1}{n}\mathbb{E}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_{ik})^{-1}}.$$

First we have

$$\begin{aligned} & \mathbb{E} \left| \text{tr} \mathbf{E}_i \mathbf{B}(z_1) r \mathbf{E}_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} \right| \\ & \leq \sum_{k \neq i} \mathbb{E} \left| \mathbb{E}_i \text{tr}(\beta_{k(i)}(z_1) - b_i(z_1)) \left( \frac{n-1}{n} b_j(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \right| \\ & \leq \sum_{k \neq i} \mathbb{E} \left| \mathbb{E}_i (\beta_{k(i)}(z) - b_i(z)) \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \boldsymbol{\alpha}_k \right| \\ & \leq \sum_{k \neq i} \mathbb{E}^{1/2} |(\beta_{k(i)}(z) - b_i(z))^2| \cdot \mathbb{E}^{1/2} \left| \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \boldsymbol{\alpha}_k \right| \\ & = \sqrt{n} \mathbb{E}^{1/2} \left| \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \boldsymbol{\alpha}_k \right| \quad (\text{by (1.15) of Bai and Silverstein (2004)}) \\ & = O(n^{1/2}) \end{aligned}$$

where  $\check{\mathbf{S}}_i$  is the analogue for the matrix  $\mathbf{S}_i$  with vectors  $\mathbf{x}_{j+1}, \dots, \mathbf{x}_n$  replaced by their iid copies  $\check{\mathbf{x}}_{j+1}, \dots, \check{\mathbf{x}}_n$ .

Second we have

$$\begin{aligned} & \frac{1}{n} \sum_{k \neq i} \mathbb{E} \left| \text{tr} \mathbf{E}_i \mathbf{C}(z_1) \cdot \mathbf{E}_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} \right| \\ & \leq \frac{1}{n} \sum_{k \neq i} \mathbb{E} \left| \text{tr} \mathbf{E}_i b_i(z) \left( \frac{n-1}{n} b_j(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \left( \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \right) \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \right| \\ & = \frac{1}{n} \sum_{k \neq i} \mathbb{E} \left| \mathbf{E}_i b_i(z) \left( \frac{n-1}{n} b_j(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \beta_{k(i)} \boldsymbol{\alpha}'_k \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \boldsymbol{\alpha}_k \right| \\ & \leq \frac{1}{n} \sum_{k \neq i} \mathbb{E} \left| \boldsymbol{\alpha}'_k \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \boldsymbol{\alpha}_k \right| \\ & \quad (\text{by (1.15) of Bai and Silverstein (2004)}) \\ & \leq K \end{aligned}$$

Third, we consider

$$\begin{aligned}
& b_i(z_1) \text{tr} E_i \mathbf{A}(z_1) E_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} \\
&= b_i(z_1) \text{tr} \sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} (\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I}) E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} \\
&= b_i(z_1) \underbrace{\sum_{k < i} \boldsymbol{\alpha}_k^* E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left[ \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} - \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right] \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k}_{\mathbf{C}_1} \\
&\quad - b_i(z_1) \text{tr} \underbrace{\frac{1}{n} \sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left[ \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} - \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right]}_{\mathbf{C}_2} \\
&\quad + b_i(z_1) \text{tr} \underbrace{\sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} (\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I}) r E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1}}_{\mathbf{C}_3}
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}|\mathbf{C}_2| &= \mathbb{E} \left| b_i(z_1) \text{tr} \frac{1}{n} \sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} r E_i \left[ \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} - \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right] \right| \\
&= \mathbb{E} \left| b_i(z_1) \frac{1}{n} \sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} E_i \beta_{k(i)} \boldsymbol{\alpha}'_k \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_1} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \boldsymbol{\alpha}_k \right| \\
&\leq K \quad (\text{similar to the proof of (??)})
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}|\mathbf{C}_3| &= \mathbb{E} \left| b_i(z_1) \text{tr} \sum_{k < i} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} (\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I}) E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right| \\
&\leq \sum_{k < i} K \mathbb{E} \left| \boldsymbol{\alpha}_k^* E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \boldsymbol{\alpha}_k - \frac{1}{n} \text{tr} E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \right| \\
&\leq \sum_{k < i} K \mathbb{E}^{\frac{1}{2}} \left| \boldsymbol{\alpha}_k^* E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \boldsymbol{\alpha}_k - \frac{1}{n} \text{tr} E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \right|^2 \\
&= O(n^{\frac{1}{2}}) \quad (\text{by (1.15) of Bai and Silverstein (2004)})
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_1 &= b_i(z_1) \sum_{k < i} \boldsymbol{\alpha}_k^* E_i \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} E_i \left[ \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_i \right)^{-1} - \left( \frac{1}{z_2} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \right] \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k \\
&= b_i(z_1) \sum_{k < i} E_i \check{\beta}_{k(i)}(z_2) \cdot \boldsymbol{\alpha}_k^* \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \boldsymbol{\alpha}_k \\
&\quad \cdot \boldsymbol{\alpha}_k^* \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k \\
&= b_i(z_1) b_i(z_2) \sum_{k < i} E_i \boldsymbol{\alpha}_k^* \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \boldsymbol{\alpha}_k \\
&\quad \cdot \boldsymbol{\alpha}_k^* \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} \boldsymbol{\alpha}_k + O_p(n^{1/2}) \quad (\text{by (4.7) and (4.8)}) \\
&= \frac{1}{n^2} b_i(z_1) b_i(z_2) \sum_{k < i} E_i \text{tr} \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \\
&\quad \cdot \text{tr} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_{ik} \right)^{-1} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} + O_p(n^{1/2}) \\
&= \frac{i-1}{n^2} b_i(z_1) b_i(z_2) E_i \text{tr} \left( \frac{1}{z_1} \mathbf{T} - \mathbf{S}_i \right)^{-1} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \\
&\quad \cdot \text{tr} \left( \frac{1}{z_2} \mathbf{T} - \check{\mathbf{S}}_i \right)^{-1} \left( \frac{n-1}{n} b_i(z_1) \mathbf{I} - \frac{1}{z_1} \mathbf{T} \right)^{-1} + O_p(n^{1/2})
\end{aligned}$$

That is,

$$\begin{aligned} & \text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} \left[ 1 + \frac{(i-1)}{n^2}b_i(z_1)b_i(z_2) \cdot \text{tr}(\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1}(\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \right] \\ = & -\text{tr} \left[ (\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \cdot \mathbf{E}_i(\frac{1}{z_2}\mathbf{T} - \mathbf{S}_i)^{-1} \right] + O_p(n^{1/2}). \end{aligned}$$

Then by (4.6) we have

$$\begin{aligned} & \text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} \\ & \left[ 1 - \frac{(i-1)}{n^2}b_i(z_1)b_i(z_2)\text{tr}(\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \cdot (\frac{n-1}{n}b_i(z_2)\mathbf{I} - \frac{1}{z_2}\mathbf{T})^{-1} \right] \\ = & \text{tr} \left[ (\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \cdot (\frac{n-1}{n}b_i(z_2)\mathbf{I} - \frac{1}{z_2}\mathbf{T})^{-1} \right] + O_p(n^{1/2}) \end{aligned}$$

because  $\text{tr}\mathbf{A}(z_1)(\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} = O_p(n^{1/2})$ ,  $\text{tr}\mathbf{B}(z_1)(\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} = O_p(n^{1/2})$  and  $\text{tr}\mathbf{C}(z_1)(\frac{n-1}{n}b_i(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} = O_p(n^{1/2})$ .

By Lemma 4.1 and 4.2, we have

$$|b_i(z) - b(z)| \leq Kn^{-1}, \quad |b_i(z) - \mathbf{E}\beta_i(z)| \leq Kn^{-1/2},$$

$$\frac{1}{pz} \sum_{i=1}^n \mathbf{E}(-1 + \beta_i(z)) = \mathbf{E}s_n(z), \quad |\mathbf{E}s_n(z) - s_n^0(z)| \leq Kn^{-1}$$

$$\mathbf{E}\beta_i(z) = y_n z \mathbf{E}s_n(z) + 1$$

So we have

$$\begin{aligned} & \text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} \\ & \left[ 1 - \frac{(i-1)}{n^2}b(z_1)b(z_2)\text{tr}(b(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \cdot (b(z_2)\mathbf{I} - \frac{1}{z_2}\mathbf{T})^{-1} \right] \\ = & \text{tr} \left[ b(z_1)\mathbf{I} - \frac{1}{z_1}\mathbf{T})^{-1} \cdot (b(z_2)\mathbf{I} - \frac{1}{z_2}\mathbf{T})^{-1} \right] + O_p(n^{1/2}) \end{aligned}$$

So we obtain

$$\begin{aligned} & b(z_1)b(z_2)\text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} \left[ 1 - \frac{(i-1)}{n}y_n \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH_n(t) \right] \\ = & p \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH_n(t) + O_p(n^{1/2}) \end{aligned}$$

That is,

$$b(z_1)b(z_2)\text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} = \frac{p \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH_n(t)}{1 - \frac{(i-1)}{n}y_n \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH_n(t)}$$

Moreover, we have

$$\begin{aligned}
& \frac{b(z_1)b(z_2)}{n^2} \sum_{i=1}^n \text{tr}[\mathbf{E}_i(\frac{1}{z_1}\mathbf{T} - \mathbf{S}_i)^{-1}](\frac{1}{z_2}\mathbf{T} - \check{\mathbf{S}}_i)^{-1} \rightarrow \int_0^1 \frac{y \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH(t)}{1 - x \cdot y \int \frac{b(z_1)b(z_2)}{(b(z_1) - \frac{1}{z_1}t) \cdot (b(z_2) - \frac{1}{z_2}t)} dH(t)} dx \\
& = a(z_1, z_2) \int_0^1 \frac{1}{1 - xa(z_1, z_2)} dx = \int_0^{a(z_1, z_2)} \frac{1}{1 - z} dz
\end{aligned}$$

where

$$\begin{aligned}
a(z_1, z_2) &= y \int \frac{b(z_1)b(z_2)}{(\frac{t}{z_1} - b(z_1)) \cdot (\frac{t}{z_2} - b(z_2))} dH(t) \\
&= y \int \frac{(1 + yz_1s(z_1)) \cdot (1 + yz_2s(z_2))}{(\frac{t}{z_1} - 1 - yz_1s(z_1)) \cdot (\frac{t}{z_2} - 1 - yz_2s(z_2))} dH(t) \\
&= \frac{y}{z_1z_2} \int \frac{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})}{(\frac{t}{z_1} + \frac{\underline{\tilde{m}}(\frac{1}{z_1})}{z_1}) \cdot (\frac{t}{z_2} + \frac{\underline{\tilde{m}}(\frac{1}{z_2})}{z_2})} dH(t) \\
&= y \int \frac{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})}{(t + \underline{\tilde{m}}(\frac{1}{z_1})) \cdot (t + \underline{\tilde{m}}(\frac{1}{z_2}))} dH(t) \\
&= \frac{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})}{\underline{\tilde{m}}(\frac{1}{z_2}) - \underline{\tilde{m}}(\frac{1}{z_1})} \cdot y \cdot \left( \int \frac{1}{t + \underline{\tilde{m}}(\frac{1}{z_1})} dH(t) - \int \frac{1}{t + \underline{\tilde{m}}(\frac{1}{z_2})} dH(t) \right) \\
&= \frac{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})}{\underline{\tilde{m}}(\frac{1}{z_2}) - \underline{\tilde{m}}(\frac{1}{z_1})} \left( \frac{1}{z_1} - \frac{1}{z_2} + \frac{\underline{\tilde{m}}(\frac{1}{z_2}) - \underline{\tilde{m}}(\frac{1}{z_1})}{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})} \right) \\
&= 1 + \frac{\underline{\tilde{m}}(\frac{1}{z_1}) \cdot \underline{\tilde{m}}(\frac{1}{z_2})}{\underline{\tilde{m}}(\frac{1}{z_2}) - \underline{\tilde{m}}(\frac{1}{z_1})} \left( \frac{1}{z_1} - \frac{1}{z_2} \right)
\end{aligned}$$

with  $\underline{\tilde{m}}(\frac{1}{z}) \triangleq -z(1 + yzs(z))$  and

$$\begin{aligned}
\int \frac{1}{t + \underline{\tilde{m}}(\frac{1}{z})} dH(t) &= \frac{1}{z} \int \frac{1}{\frac{t}{z} + \frac{1}{z} \underline{\tilde{m}}(\frac{1}{z})} dH(t) \\
&= \frac{1}{z} \int \frac{1}{\frac{t}{z} - (1 + yzs(z))} dH(t) = \frac{\tilde{m}(z)}{z} \quad (\text{by (4.20) and (4.25)}) \\
&= \frac{s(z)}{1 + yzs(z)} = \frac{1}{y} \left( \frac{1}{z} + \frac{1}{\underline{\tilde{m}}(\frac{1}{z})} \right) \quad (\text{by (4.25)}).
\end{aligned}$$

That is, the covariance is

$$\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \int_0^{a(z_1, z_2)} \frac{1}{1 - z} dz = \frac{\partial}{\partial z_2} \left( \frac{\partial a(z_1, z_2) / \partial z_1}{1 - a(z_1, z_2)} \right)$$

$$\begin{aligned}
\partial a(z_1, z_2)/\partial z_1 &= \left( \frac{\tilde{m}(\frac{1}{z_1}) \cdot \tilde{m}(\frac{1}{z_2})}{\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})} \left( \frac{1}{z_1} - \frac{1}{z_2} \right) \right)_{z_1} \\
&= - \frac{(\tilde{m}(\frac{1}{z_1}))' \tilde{m}(\frac{1}{z_2}) (\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})) + \tilde{m}(\frac{1}{z_1})' \tilde{m}(\frac{1}{z_1}) \tilde{m}(\frac{1}{z_2})}{(\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))^2} \left( \frac{1}{z_1} - \frac{1}{z_2} \right) \\
&\quad + \frac{\tilde{m}(\frac{1}{z_1}) \cdot \tilde{m}(\frac{1}{z_2})}{\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})} \left( \frac{-1}{z_1^2} \right) \\
&= - \frac{(\tilde{m}(\frac{1}{z_1}))' \tilde{m}^2(\frac{1}{z_2})}{(\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))^2} \left( \frac{1}{z_1} - \frac{1}{z_2} \right) + \frac{\tilde{m}(\frac{1}{z_1}) \cdot \tilde{m}(\frac{1}{z_2})}{\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})} \left( \frac{-1}{z_1^2} \right).
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\frac{\partial a(z_1, z_2)/\partial z_1}{1 - a(z_1, z_2)} \\
&= \left[ - \frac{(\tilde{m}(\frac{1}{z_1}))' \tilde{m}^2(\frac{1}{z_2})}{(\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))^2} \left( \frac{1}{z_1} - \frac{1}{z_2} \right) + \frac{\tilde{m}(\frac{1}{z_1}) \cdot \tilde{m}(\frac{1}{z_2})}{\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})} \left( \frac{-1}{z_1^2} \right) \right] \frac{\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1})}{\tilde{m}(\frac{1}{z_1}) \tilde{m}(\frac{1}{z_2}) \left( \frac{1}{z_1} - \frac{1}{z_2} \right)} \\
&= - \frac{(\tilde{m}(\frac{1}{z_1}))' \tilde{m}(\frac{1}{z_2})}{m(\frac{1}{z_1}) (\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))} - \frac{1/z_1^2}{1/z_1 - 1/z_2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z_2} \left( \frac{\partial a(z_1, z_2)/\partial z_1}{1 - a(z_1, z_2)} \right) &= \frac{(\tilde{m}(\frac{1}{z_1}))' (\tilde{m}(\frac{1}{z_2}))'}{(\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))^2} - \frac{1}{z_1^2 z_2^2} \frac{1}{(1/z_1 - 1/z_2)^2} \\
&= \frac{(\tilde{m}(\frac{1}{z_1}))' (\tilde{m}(\frac{1}{z_2}))'}{(\tilde{m}(\frac{1}{z_2}) - \tilde{m}(\frac{1}{z_1}))^2} - \frac{1}{(z_1 - z_2)^2} \\
&= \frac{[z_1(1 + yz_1s(z_1))]'[z_2(1 + yz_2s(z_2))]' }{[z_1(1 + yz_1s(z_1)) - z_2(1 + yz_2s(z_2))]^2} - \frac{1}{(z_1 - z_2)^2}.
\end{aligned}$$

That is, by (4.5) we have

$$\text{Cov}(M(z_1), M(z_2)) = \kappa \left( \frac{[z_1(1 + yz_1s(z_1))]'[z_2(1 + yz_2s(z_2))]' }{[z_1(1 + yz_1s(z_1)) - z_2(1 + yz_2s(z_2))]^2} - \frac{1}{(z_1 - z_2)^2} \right). \quad (4.9)$$

Then the proof of Theorem 4.1 is completed. ■

## 4.2 Tightness of $M_n^1(z)$

**Theorem 4.2** *Under Assumptions 1\*, 2\* and 3, the sequence of random functions  $M_n^1(z)$  is tight for  $z \in \mathcal{C} \cup \bar{\mathcal{C}}$ .*

PROOF. We want to show that

$$\sup_{n; z_1, z_2 \in \mathcal{C}_n} \frac{\mathbb{E}|M_n^1(z_1) - M_n^1(z_2)|^2}{|z_1 - z_2|^2}$$

is finite. It is straightforward to verify that this will be true if we can find a  $K > 0$  for which

$$\sup_{n; z_1, z_2 \in \mathcal{C}_n} \frac{\mathbb{E}|M_n^1(z_1) - M_n^1(z_2)|^2}{|z_1 - z_2|^2} \leq K.$$

$$M_n^1(z_1) - M_n^1(z_2) = p(s_n(z_1) - s_n(z_2)) - p \cdot \mathbb{E}(s_n(z_1) - s_n(z_2))$$

$$s_n(z_1) - s_n(z_2) = \frac{1}{p} \text{tr}[(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1} - (\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1}] = \frac{1}{p}(z_1 - z_2) \text{tr} \mathbf{D}^{-1}(z_1) \mathbf{D}^{-1}(z_2)$$

where  $\mathbf{D}(z) = (\mathbf{S}^{-1}\mathbf{T} - z\mathbf{I})^{-1}$ . We have

$$\begin{aligned} \frac{M_n^1(z_1) - M_n^1(z_2)}{z_1 - z_2} &= p \cdot \frac{s_n(z_1) - s_n(z_2) - \mathbb{E}(s_n(z_1) - s_n(z_2))}{z_1 - z_2} \\ &= \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \text{tr} \mathbf{D}^{-1}(z_1) \mathbf{D}^{-1}(z_2) \\ &= \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \text{tr}(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1} (\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} \end{aligned}$$

and

$$\begin{aligned} (\mathbf{S}^{-1}\mathbf{T} - z\mathbf{I})^{-1} &= \mathbf{S}(\mathbf{T} - z\mathbf{S})^{-1} \\ &= \mathbf{S}(\mathbf{T} - z\mathbf{S}_i - z\boldsymbol{\alpha}_i\boldsymbol{\alpha}_i')^{-1} \\ &= (\mathbf{S}_i + \boldsymbol{\alpha}_i\boldsymbol{\alpha}_i') \left( \mathbf{D}_i^{-1} + \frac{z \cdot \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \right) \\ &= \mathbf{S}_i \mathbf{D}_i^{-1} + \frac{z \cdot \mathbf{S}_i \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} + \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} + \frac{z \cdot \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \\ &= \mathbf{S}_i \mathbf{D}_i^{-1} + \frac{z \cdot \mathbf{S}_i \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} + \frac{\boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \\ &= (\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1} + \frac{z \cdot \mathbf{S}_i \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} + \frac{\boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \\ &= (\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1} + \frac{(z \cdot \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} \end{aligned}$$

where  $\mathbf{S}_i \mathbf{D}_i^{-1} = (\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1}$ . That is,

$$(\mathbf{S}^{-1}\mathbf{T} - z\mathbf{I})^{-1} - (\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1} = \frac{(z \cdot \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1}}{1 - z \cdot \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \boldsymbol{\alpha}_i} = (z \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \mathbf{D}_i^{-1} \beta_i(z).$$

$$\begin{aligned}
& \text{tr}(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}(\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} - \text{tr}(\mathbf{S}_i^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}(\mathbf{S}_i^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} \\
= & \text{tr}[(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1} - (\mathbf{S}_i^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}][(\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} - (\mathbf{S}_i^{-1}\mathbf{T} - z_2\mathbf{I})^{-1}] \\
& + \text{tr}[(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1} - (\mathbf{S}_i^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}](\mathbf{S}_i^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} \\
& + \text{tr}(\mathbf{S}_i^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}[(\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} - (\mathbf{S}_i^{-1}\mathbf{T} - z_2\mathbf{I})^{-1}] \\
= & \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1}(z_2 \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1}(z_1 \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \cdot \beta_i(z_1) \beta_i(z_2) \\
& + \beta_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1}(\mathbf{S}_i^{-1}\mathbf{T} - z_2\mathbf{I})^{-1}(z_1 \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \\
& + \beta_i(z_2) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1}(\mathbf{S}_i^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}(z_2 \mathbf{S}_i \mathbf{D}_i^{-1} + \mathbf{I}) \boldsymbol{\alpha}_i \\
= & (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i)^2 \cdot \beta_i(z_1) \beta_i(z_2) + \beta_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \\
& + \beta_i(z_2) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_1) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i
\end{aligned}$$

where  $\mathbf{F}_i^{-1}(z) = (\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1}$  and  $(\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1} = -\frac{1}{z}\mathbf{I} + \frac{1}{z}\mathbf{S}_i^{-1}\mathbf{T}(\mathbf{S}_i^{-1}\mathbf{T} - z\mathbf{I})^{-1}$ .

$$\frac{M_n^1(z_1) - M_n^1(z_2)}{z_1 - z_2} = \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \text{tr}(\mathbf{S}^{-1}\mathbf{T} - z_1\mathbf{I})^{-1}(\mathbf{S}^{-1}\mathbf{T} - z_2\mathbf{I})^{-1} \quad (4.10)$$

$$= \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i)^2 \cdot \beta_i(z_1) \beta_i(z_2) \quad (4.11)$$

$$+ \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \beta_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \quad (4.12)$$

$$+ \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \beta_i(z_2) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_1) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i \quad (4.13)$$

Our goal is to show that the absolute second moment of (4.10) is bounded. We begin with (4.12). We have

$$\begin{aligned}
& \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \beta_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \\
= & \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) b_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \\
& - \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \beta_i(z_1) b_i(z_1) \varepsilon_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \\
= & W_1 - W_2
\end{aligned}$$

where  $\varepsilon_i(z) = \boldsymbol{\alpha}'_i (\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i - \frac{1}{n} \text{Etr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}$  and

$$\begin{aligned}
\mathbf{E}|W_1|^2 &= \mathbf{E} \left| \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) b_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \right|^2 \\
&= \sum_{i=1}^n \mathbf{E} |(\mathbf{E}_i - \mathbf{E}_{i-1}) b_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i|^2 \\
&= \sum_{i=1}^n b_i^2(z_1) \mathbf{E} | \mathbf{E}_i \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i - \frac{1}{n} \text{tr} \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) |^2 \\
&\leq K
\end{aligned}$$



by Lemma 4.2. Moreover, we have

$$\begin{aligned}
\mathbb{E}|W_2|^2 &= \sum_{i=1}^n \mathbb{E}|(\mathbf{E}_i - \mathbf{E}_{i-1})\beta_i(z_1)b_i(z_1)\varepsilon_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i|^2 \\
&= n \mathbb{E}|(\mathbf{E}_i - \mathbf{E}_{i-1})\beta_i(z_1)b_i(z_1)\varepsilon_i(z_1) \cdot \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i|^2 \\
&\leq 2nb_i^2(z_1) \cdot \mathbb{E}\beta_i^2(z_1)\varepsilon_i^2(z_1) \cdot (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i)^2 \leq K
\end{aligned}$$

where

$$\begin{aligned}
|\beta_i(z)| &= |1 - \boldsymbol{\alpha}'_i (\frac{1}{z} \mathbf{T} - \mathbf{S}_i)^{-1} \boldsymbol{\alpha}_i| \leq 1 + |z \boldsymbol{\alpha}_i \mathbf{S}_i^{-1} (\mathbf{T} \mathbf{S}_i^{-1} - z \mathbf{I})^{-1} \boldsymbol{\alpha}_i| \\
&\leq 1 + |z| \cdot |\boldsymbol{\alpha}_i|^2 \|\mathbf{S}_i^{-1} (\mathbf{T} \mathbf{S}_i^{-1} - z \mathbf{I})^{-1}\| \\
&\leq 1 + K |\boldsymbol{\alpha}_i|^2 + n^5 I(\|\mathbf{S}_i\| \geq \eta_r \text{ or } \lambda_{\min}^{\mathbf{S}_i} \leq \eta_l)
\end{aligned}$$

and

$$\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{F}_i^{-1}(z_2) \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_1) \boldsymbol{\alpha}_i \leq K |\boldsymbol{\alpha}_i|^2 + n^t I(\|\mathbf{S}_i\| \geq \eta_r \text{ or } \lambda_{\min}^{\mathbf{S}_i} \leq \eta_l).$$

Similarly, it can be obtained that the second moment of (4.13) is uniformly finite. Now we begin (4.11). We have

$$\begin{aligned}
&\sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i)^2 \cdot \beta_i(z_1) \beta_i(z_2) \\
&= \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) \left( \left[ \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i \right]^2 - \left[ \frac{1}{n} \text{tr}(\mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2)) \right]^2 \right) \cdot b_i(z_1) b_i(z_2) \\
&\quad - \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i)^2 \cdot \beta_i(z_1) \beta_i(z_2) b_i(z_2) \varepsilon_i(z_2) \\
&\quad - \sum_{i=1}^n (\mathbf{E}_i - \mathbf{E}_{i-1}) (\boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i)^2 \cdot b_i(z_1) b_i(z_2) \beta_i(z_1) \varepsilon_i(z_1) \\
&= Z_1 - Z_2 - Z_3.
\end{aligned}$$

By similar methods to  $W_2$ , we obtain that the second moments of  $Z_2$  and  $Z_3$  are uniformly finite. Now we begin  $Z_1$ . We have

$$\begin{aligned}
&\mathbb{E}|Z_1|^2 \\
&\leq K \sum_{i=1}^n \mathbb{E} \left| \left[ \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i \right]^2 - \left[ \frac{1}{n} \text{tr}(\mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2)) \right]^2 \right|^2 \\
&\leq 2K \sum_{i=1}^n \mathbb{E} \left| \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i - \frac{1}{n} \text{tr}(\mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2)) \right|^4 \\
&\quad + \frac{K}{n} \sum_{i=1}^n \mathbb{E} \left| \left( \boldsymbol{\alpha}'_i \mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2) \boldsymbol{\alpha}_i - \frac{1}{n} \text{tr}(\mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2)) \right) \cdot |\mathbf{D}_i^{-1} \mathbf{S}_i^{-1} \mathbf{T} \mathbf{F}_i^{-1}(z_2)| \right|^2 \\
&\leq K.
\end{aligned}$$

Then  $\sup_{n; z_1, z_2 \in \mathcal{C}^+} \frac{\mathbb{E}[|M_n^1(z_1) - M_n^1(z_2)|]^2}{|z_1 - z_2|^2} \leq K$ .

So the proof of Theorem 4.2 is completed. ■

### 4.3 Uniform convergence of $M_n^2(z) = p(\mathbb{E}s_n(z) - s_n^0(z))$ for $z \in \mathcal{C}$

**Theorem 4.3** *We have*

$$\sup_{z \in \mathcal{C}_n} \left| M_n^2(z) - \frac{1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

where  $M_n^2(z)$  converges uniformly to the limit

$$\frac{1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} = \frac{1}{2} \frac{d \log \left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)}{dz}.$$

PROOF. We have

$$\frac{1}{p} \text{tr}(\mathbf{S}^{-1} \mathbf{T} - z \mathbf{I})^{-1} = -\frac{1}{z} - \frac{1}{z^2} \cdot \frac{1}{p} \text{tr} \left( \mathbf{S} - \frac{\mathbf{T}}{z} \right)^{-1} \mathbf{T}$$

and

$$s_n(z) = \frac{1}{pz} \sum_{i=1}^n \frac{\boldsymbol{\alpha}'_i \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \boldsymbol{\alpha}_i}{1 - \boldsymbol{\alpha}'_i \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \boldsymbol{\alpha}_i} = -\frac{1}{yz} + \frac{1}{yz} \cdot \frac{1}{n} \sum_{i=1}^n \frac{1}{1 - \boldsymbol{\alpha}'_i \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \boldsymbol{\alpha}_i}.$$

By Lemma 4.4, we have  $s(z) = -\frac{1}{z} - \frac{1}{z^2} \cdot \tilde{s}(z)$  where

$$\tilde{s}(z) = \frac{-z}{1 - y\tilde{m}(z)} = \frac{-z}{1 - y \int \frac{1}{\frac{t}{z} - \frac{1}{1 - y\tilde{m}(z)}} dH(t)} = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \tilde{s}(z)} dH(t)},$$

$\tilde{m}(z)$  is the limit of  $\frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1}$ ,  $\tilde{s}(z)$  is the limit of  $\frac{1}{p} \text{tr} \left( \mathbf{S} - \frac{\mathbf{T}}{z} \right)^{-1} \mathbf{T}$  and  $\tilde{s}(z) = -(1 - y)z + y\tilde{s}(z)$ . We have

$$\mathbb{E}\tilde{s}_n(z) = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \mathbb{E}\tilde{s}_n(z)} dH_n(t) - R_n}$$

where

$$R_n = -\frac{1}{z} + y \int \frac{1}{t + \mathbb{E}\tilde{s}_n(z)} dH_n(t) - \frac{1}{\mathbb{E}\tilde{s}_n(z)}.$$

Moreover, let

$$\tilde{s}_n^0(z) = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \tilde{s}_n^0(z)} dH_n(t)}$$

and

$$s_n^0(z) = -\frac{1}{z} - \frac{1}{z^2} \cdot \tilde{s}_n^0(z), \quad \tilde{s}_n^0(z) = -(1-y)z + y\tilde{s}_n^0(z)$$

where  $s_n^0(z)$  is the Stieltjes transform of  $F^{y_n, H_n}$ . So we obtain

$$E\tilde{s}_n(z) - \tilde{s}_n^0(z) = \frac{(E\tilde{s}_n(z) - \tilde{s}_n^0(z))y \int \frac{1}{(t+E\tilde{s}_n(z))(t+\tilde{s}_n^0(z))} dH_n(t)}{\left(-\frac{1}{z} + y \int \frac{1}{t+E\tilde{s}_n(z)} dH_n(t) - R_n\right) \left(-\frac{1}{z} + y \int \frac{1}{t+\tilde{s}_n^0(z)} dH_n(t)\right)} + E\tilde{s}_n(z)\tilde{s}_n^0(z)R_n$$

That is,

$$n(E\tilde{s}_n(z) - \tilde{s}_n^0(z)) = \frac{E\tilde{s}_n(z)\tilde{s}_n^0(z)}{1 - \frac{y \int \frac{1}{(t+E\tilde{s}_n(z))(t+\tilde{s}_n^0(z))} dH_n(t)}{\left(-\frac{1}{z} + y \int \frac{1}{t+E\tilde{s}_n(z)} dH_n(t) - R_n\right) \left(-\frac{1}{z} + y \int \frac{1}{t+\tilde{s}_n^0(z)} dH_n(t)\right)}} \cdot nR_n$$

$$R_n = -\frac{1}{E\tilde{s}_n(z)} \cdot \left( \frac{y}{z} E\tilde{s}_n(z) + y \int \frac{tdH_n(t)}{t + E\tilde{s}_n(z)} \right) = -\frac{1}{E\tilde{s}_n(z)} \cdot \frac{y}{z} \left( E\tilde{s}_n(z) + \int \frac{tdH_n(t)}{t/z + E\tilde{s}_n(z)/z} \right)$$

$$\left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right) = \left( \frac{1}{z} \mathbf{T} + \frac{E\tilde{s}_n(z)}{z} \mathbf{I} \right) - \frac{E\tilde{s}_n(z)}{z} \mathbf{I} - \sum_{i=1}^n \alpha_i \alpha_i'$$

So we obtain

$$\begin{aligned} & \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\ &= \left( \frac{1}{z} \mathbf{T} - E\beta_i(z) \mathbf{I} \right)^{-1} \mathbf{T} + \left( \frac{1}{z} \mathbf{T} - E\beta_i(z) \mathbf{I} \right)^{-1} \left( \sum_{i=1}^n \alpha_i \alpha_i' - E\beta_i(z) \mathbf{T} \right) \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\ &= \left( \frac{1}{z} \mathbf{T} - E\beta_i(z) \mathbf{I} \right)^{-1} \mathbf{T} - E\beta_i(z) \cdot \left( \frac{1}{z} \mathbf{T} - E\beta_i(z) \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\ & \quad + \sum_{i=1}^n \left( \frac{1}{z} \mathbf{T} - E\beta_i(z) \mathbf{I} \right)^{-1} \alpha_i \alpha_i' \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \mathbf{T} \beta_i(z) \end{aligned}$$

where  $\beta_i(z) = \frac{1}{1 - \alpha'_i(\frac{\mathbf{T}}{z} - \mathbf{S}_i)^{-1}\alpha_i}$ . Taking expected values and trace on both sides and dividing by  $p$ , we get

$$\begin{aligned}
& \frac{1}{p} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\
&= \frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \mathbf{T} - \text{E}\beta_1(z) \cdot \frac{1}{p} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\
&\quad + \frac{1}{p} \text{Etr} \sum_{i=1}^n \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \alpha_i \alpha'_i \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_i \right)^{-1} \mathbf{T} \beta_i(z) \\
&= \frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \mathbf{T} - \text{E}\beta_1(z) \cdot \frac{1}{p} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\
&\quad + \frac{1}{y_n} \text{E}\beta_1(z) \alpha'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \alpha_1 \\
&= \frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \mathbf{T} + \frac{1}{y_n} \text{E}\beta_1(z) \left[ \alpha'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \alpha_1 \right. \\
&\quad \left. - \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \\
&\quad + \left[ - \frac{1}{y_n} \text{E}\beta_1(z) \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \right. \\
&\quad \left. + \frac{1}{y_n} \text{E}\beta_1(z) \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \right] + o\left(\frac{1}{n}\right) \\
&= \frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \mathbf{T} + \frac{1}{y_n} \text{E}\beta_1(z) \left[ \alpha'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \alpha_1 \right. \\
&\quad \left. - \frac{1}{n} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \\
&\quad - \frac{(\text{E}\beta_1(z))^2}{y} \frac{1}{n^2} \text{Etr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \text{E}\beta_1(z) \mathbf{I} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} + o\left(\frac{1}{n}\right)
\end{aligned}$$

So we obtain

$$\begin{aligned}
\frac{-z\mathbb{E}\tilde{s}_n(z)}{y} \cdot R_n &= \int \frac{tdH_n(t)}{\frac{t}{z} - \mathbb{E}\beta_1(z)} - \mathbb{E}\tilde{s}_n(z) \\
&= \frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \cdot \mathbf{I} \right)^{-1} \mathbf{T} - \frac{1}{p} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \mathbf{T} \\
&= -\frac{1}{y_n} \mathbb{E}\beta_1(z) \left[ \boldsymbol{\alpha}'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \boldsymbol{\alpha}_1 \right. \\
&\quad \left. - \frac{1}{n} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \\
&\quad + \frac{(\mathbb{E}\beta_1(z))^2}{y} \frac{1}{n^2} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} + o\left(\frac{1}{n}\right) \\
&= -\frac{1}{y_n} \mathbb{E}\bar{\beta}_1^2(z) \left[ \boldsymbol{\alpha}'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \boldsymbol{\alpha}_1 \right. \\
&\quad \left. - \frac{1}{n} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \varepsilon_j \\
&\quad + \frac{(\mathbb{E}\beta_1(z))^2}{y} \frac{1}{n^2} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} + o\left(\frac{1}{n}\right) \\
&= -\frac{(\mathbb{E}\beta_1(z))^2}{y} \cdot \mathbb{E} \left[ \boldsymbol{\alpha}'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \boldsymbol{\alpha}_1 \right. \\
&\quad \left. - \frac{1}{n} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \varepsilon_j \\
&\quad + \frac{(\mathbb{E}\beta_1(z))^2}{y} \frac{1}{n^2} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} + o\left(\frac{1}{n}\right) \\
&= -\frac{(\mathbb{E}\beta_1(z))^2}{y} \cdot \mathbb{E} \left[ \boldsymbol{\alpha}'_1 \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \boldsymbol{\alpha}_1 \right. \\
&\quad \left. - \frac{1}{n} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} \right] \varepsilon_j \\
&\quad + \frac{(\mathbb{E}\beta_1(z))^2}{y \cdot n^2} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - \mathbb{E}\beta_1(z) \mathbf{I} \right)^{-1} + o\left(\frac{1}{n}\right) \quad (4.14)
\end{aligned}$$

where  $\beta_j = \bar{\beta}_j + \bar{\beta}_j^2 \varepsilon_j + \bar{\beta}_j^2 \beta_j \varepsilon_j^2$ ,  $\varepsilon_j = \boldsymbol{\alpha}'_j \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_j \right)^{-1} \boldsymbol{\alpha}_j - \frac{1}{n} \mathbb{E} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_j \right)^{-1}$  and  $\beta_j(z) = \frac{1}{1 - ym(z)} + O\left(\frac{1}{n}\right)$ .

By (1.15) of Bai and Silverstein (2004) and (4.14), when all  $x_{tj}$  are complex

$$R_n = 0. \quad (4.15)$$

In RSE case,

$$\begin{aligned}
& \frac{E\tilde{s}_n(z)z}{y} \cdot R_n \\
&= \frac{-(E\beta_1(z))^2}{y} \frac{1}{n^2} E\text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} + o\left(\frac{1}{n}\right) \\
&= \frac{-(E\beta_1(z))^2}{y} \frac{1}{n^2} E\text{tr} \left\{ \begin{aligned} & -\mathbf{T} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \\ & -E\beta_1(z) \cdot \sum_{k \neq 1} E_2^{-1}(z) \frac{1}{n} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_1^{-1}(z) \left( \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \right) \\ & +E\beta_1(z) \cdot \sum_{k \neq 1} E_2^{-1}(z) \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_1^{-1}(z) \left( \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \right) \\ & +E\beta_1(z) \cdot \sum_{k \neq 1} E_2^{-1}(z) \left( \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I} \right) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_1^{-1}(z) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \end{aligned} \right\} + O\left(\frac{1}{n^{3/2}}\right) \\
&= \frac{(E\beta_1(z))^2}{yn^2} E\text{tr} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \mathbf{T} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \\
&\quad + \frac{E\beta_1(z)(E\beta_1(z))^2}{yn^2} E\text{tr} \sum_{k \neq 1} E_2^{-1}(z) \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \frac{\left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_1^{-1}(z) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1}}{1 - \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \boldsymbol{\alpha}_k} + O\left(\frac{1}{n^{3/2}}\right) \\
&= \frac{(E\beta_1(z))^2}{yn^2} E\text{tr} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \mathbf{T} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} \\
&\quad + y^{-1} n^{-2} (E\beta_1(z))^3 \sum_{k \neq 1} E \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_2^{-1}(z) \boldsymbol{\alpha}_k \cdot \frac{\boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-2} \mathbf{T} E_1^{-1}(z) \boldsymbol{\alpha}_k}{1 - \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \boldsymbol{\alpha}_k} + O\left(\frac{1}{n^{3/2}}\right) \\
&= -\frac{(E\beta_1(z))^2}{y} \frac{1}{n^2} E\text{tr} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \mathbf{T} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-2} \\
&\quad + \frac{(E\beta_1(z))^4}{yn^2} \sum_{k \neq 1} E \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \mathbf{T} E_2^{-1}(z) \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_k^* \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-2} \mathbf{T} E_1^{-1}(z) \boldsymbol{\alpha}_k + O\left(\frac{1}{n^{3/2}}\right)
\end{aligned}$$

where  $E_1^{-1}(z) = \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1}$ ,  $E_2^{-1}(z) = \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1}$  and  $\mathbf{A}(z) = \sum_{k \neq i} \left( \frac{n-1}{n} b_i(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} (\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k^* - \frac{1}{n} \mathbf{I}) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{ik} \right)^{-1}$  because

$$\begin{aligned}
& -\frac{-1}{y(1-y\bar{m}(z))^2} \frac{1}{n^2} E\text{tr} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \sum_{k \neq 1} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \frac{1}{n} \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \\
& \cdot \left( \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_1 \right)^{-1} - \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-1} \right) = O\left(\frac{1}{n^2}\right)
\end{aligned}$$

and

$$\begin{aligned} & \frac{-1}{y(1-y\tilde{m}(z))^2} \frac{1}{n^2} E \text{tr} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1} \sum_{k \neq 1} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-1} \\ & \cdot \left( \alpha_k \alpha_k^* - \frac{1}{n} \mathbf{I} \right) \left( \frac{1}{z} \mathbf{T} - \mathbf{S}_{1k} \right)^{-2} = O\left(\frac{1}{n^{3/2}}\right). \end{aligned}$$

$$\begin{aligned} & \frac{-z E \tilde{\underline{s}}_n(z)}{y} \cdot p R_n = -z E \tilde{\underline{s}}_n(z) \cdot n R_n \\ & = - \frac{\frac{(E\beta_1(z))^2}{y} \cdot \frac{y}{n} E \text{tr} (E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T})^{-2} \mathbf{T} \left( \frac{1}{z} \mathbf{T} - E\beta_1(z) \mathbf{I} \right)^{-1}}{1 - \frac{1}{(1-y\tilde{m}(z))^2} \cdot \frac{1}{n} \text{tr} \left( E\beta_1(z) \mathbf{I} - \frac{1}{z} \mathbf{T} \right)^{-2}} \\ & = - \frac{\frac{1}{(1-y\tilde{m}(z))^2} \cdot \int \frac{y t dH(t)}{(E\beta_1(z) - \frac{t}{z})^2 (\frac{t}{z} - E\beta_1(z))}}{1 - \frac{y}{(1-y\tilde{m}(z))^2} \int \frac{dH(t)}{(E\beta_1(z) - \frac{t}{z})^2}} + o(1) \\ & = - \frac{\frac{1}{(1-y\tilde{m}(z))^2} \cdot \int \frac{y t dH(t)}{(t/z - 1/(1-y\tilde{m}(z)))^3}}{1 - \frac{y}{(1-y\tilde{m}(z))^2} \cdot \int \frac{dH(t)}{(t/z - 1/(1-y\tilde{m}(z)))^2}} + o(1) \\ & = - \frac{y \int \frac{t(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^3}}{1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}} \cdot + o(1) \end{aligned}$$

where  $E\beta_1(z) \rightarrow \frac{1}{1-y\tilde{m}(z)}$  by (4.26). Thus by (4.15), (4.22) and (4.25), we have

$$\begin{aligned} p(Es_n(z) - s_n^0(z)) &= \frac{-1}{z^2} \cdot p(E\tilde{s}_n(z) - \tilde{s}_n^0(z)) \\ &= \frac{-1}{z^2} \cdot n(E\tilde{\underline{s}}_n(z) - \tilde{\underline{s}}_n^0(z)) \\ &= \frac{-1}{z^2} \cdot \frac{E\tilde{\underline{s}}_n(z) \tilde{\underline{s}}_n^0(z)}{1 - \frac{y \int \frac{1}{(t+E\tilde{\underline{s}}_n(z))(t+\tilde{\underline{s}}_n^0(z))} dH_n(t)}{\left(-\frac{1}{z} + y \int \frac{1}{t+E\tilde{\underline{s}}_n(z)} dH_n(t) - R_n\right) \left(-\frac{1}{z} + y \int \frac{1}{t+\tilde{\underline{s}}_n^0(z)} dH_n(t)\right)}} \cdot n R_n \\ &= \frac{\kappa - 1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} + o(1) \end{aligned} \tag{4.16}$$

So we conclude that in the RSE case

$$\sup_{z \in \mathcal{C}_n} \left| M_n^2(z) - \frac{\kappa - 1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z - 1 - yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z - 1 - yzs(z))^2}\right)^2} \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

By (4.31), we obtain

$$EM(z) = \frac{\kappa - 1}{z^2} \cdot \frac{y \int \frac{t(1+yzs(z))^3 dH(t)}{(t/z-1-yzs(z))^3}}{\left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z-1-yzs(z))^2}\right)^2} = \frac{-(\kappa - 1)}{z^2} \cdot \frac{y \int \frac{t(-z)^3(1+yzs(z))^3 dH(t)}{(t-z-yz^2s(z))^3}}{\left(1 - y \int \frac{z^2(1+yzs(z))^2 dH(t)}{(t-z-yz^2s(z))^2}\right)^2}$$

That is,

$$EM(z) = \frac{\kappa - 1}{2} \frac{d \log \left(1 - y \int \frac{(1+yzs(z))^2 dH(t)}{(t/z-1-yzs(z))^2}\right)}{dz}$$

So the proof of Theorem 4.3 is completed. ■

## 4.4 Some Notations and Lemmas

**Lemma 4.1** (*Bai and Silverstein (2010) P225*) Suppose that  $x_i$ ,  $i = 1, \dots, n$  are independent, with  $Ex_i = 0$ ,  $E|x_i|^2 = 1$ ,  $\sup E|x_i|^4 = \nu < +\infty$  and  $|x_i| \leq \eta_n \sqrt{n}$  with  $\eta_n > 0$ . Assume that  $\mathbf{A}$  is a complex matrix. Then, for any given  $2 \leq \ell \leq b \log(n\nu^{-1}\eta_n^4)$  and  $b > 1$ , we have

$$E|\boldsymbol{\alpha}^* \mathbf{A} \boldsymbol{\alpha} - \text{tr}(\mathbf{A})|^l \leq \nu n^l (n\eta_n^4)^{-1} (40b^2 \|\mathbf{A}\| \eta_n^2)^l$$

where  $\boldsymbol{\alpha} = (x_1, \dots, x_n)^T$ .

**Lemma 4.2** (*Bai and Silverstein (2010) P271*) We have

$$\left| E \left( \prod_{k=1}^m \gamma_t^* \mathbf{A}_k \gamma_t \right) \prod_{l=1}^q (\gamma_t^* \mathbf{B}_l \gamma_t - n^{-1} \text{tr} \mathbf{T} \mathbf{B}_l) \right| \leq K n^{-(1 \wedge q)} \eta_n^{(2q-4) \vee 0} \prod_{k=1}^m \|\mathbf{A}_k\| \prod_{l=1}^q \|\mathbf{B}_l\|,$$

where  $m \geq 0$ ,  $q \geq 0$ ,  $\gamma_t = \frac{1}{\sqrt{n}} \mathbf{T}^{\frac{1}{2}} \mathbf{X}_t$ ,  $\mathbf{X}_t = (x_{t1}, \dots, x_{tp})^T$ ,  $(x_{tj}, t = 1, \dots, n, j = 1, \dots, p)$  are independent with  $Ex_{tj} = 0$ ,  $E|x_{tj}|^2 = 1$ ,  $\sup E|x_{tj}|^4 = \nu < +\infty$  and  $|x_{tj}| \leq \eta_n \sqrt{n}$  with  $\eta_n > 0$ .

**Lemma 4.3** Under Assumptions 1-2, we obtain

$$\frac{1}{p} \text{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} \rightarrow \tilde{m}(z), \text{ a.s.} \quad (4.17)$$

where  $\tilde{m}(z)$  is the unique solution to the equation  $\tilde{m}(z) = \int \frac{dH(t)}{-\frac{t}{z} - \frac{1}{1-y\tilde{m}(z)}}$  satisfying

$$\Im(z) \cdot \Im(\tilde{m}(z)) \geq 0.$$



Proof. For any real  $z < 0$  and complex  $w$  with  $\Im(w) > 0$ , by (4.1.2) of Page 61 of Bai and Silverstein (2010), we have

$$\begin{aligned} \frac{1}{p} \operatorname{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} + w \mathbf{I} \right)^{-1} &= -\frac{1}{p} \operatorname{tr} \left( \frac{1}{-z} \mathbf{T} + \mathbf{S} - w \mathbf{I} \right)^{-1} \\ \rightarrow -\tilde{m}(z, w) &= -m_{H_z} \left( w - \frac{1}{y} \int \frac{\tau dH_0(\tau)}{1 + \tau \tilde{m}(z, w)} \right), \quad a.s. \end{aligned} \quad (4.18)$$

where  $\tilde{m}(z, w)$  is limit of the Stieltjes transform of the matrix  $\frac{1}{-z} \mathbf{T} + \mathbf{S}$ ,  $m_{H_z}$  is the Stieltjes transform of  $H_z$ , the LSD of  $\frac{1}{-z} \mathbf{T}$ , and  $H_0(\tau) = I_{(\tau > y)}$ . By Theorem 5.11 and Lemma 2.14 (Vitali Lemma) of Bai and Silverstein (2010), the convergence of (4.18) is also true for  $w = 0$ . That is,

$$\begin{aligned} \frac{1}{p} \operatorname{tr} \left( \frac{1}{z} \mathbf{T} - \mathbf{S} \right)^{-1} &\rightarrow -\tilde{m}(z, 0) = -m_{H_z} \left( -\frac{1}{1 + y \tilde{m}(z, 0)} \right), \quad a.s. \\ &= - \int \frac{1}{\lambda - \frac{1}{1 - y \tilde{m}(z, 0)}} dH_z(\lambda) = - \int \frac{1}{\frac{\lambda}{-z} + \frac{1}{1 + y \tilde{m}(z, 0)}} dH(\lambda), \quad a.s. \end{aligned} \quad (4.19)$$

Denoting  $\tilde{m}(z) = -\tilde{m}(z, 0)$ , then the convergence of (4.17) is proved for all real nonpositive  $z$ . Noting that both sides of (4.17) are analytic functions of  $z$  on the region  $D^- = \{z \in \mathbb{C} : z \text{ is not nonpositive real number}\}$ , applying Vitali Lemma again, we conclude that (4.17) is true for all  $z \in D^-$  and  $\tilde{m}(z)$  satisfies

$$\tilde{m}(z) = \int \frac{1}{\frac{\lambda}{z} - \frac{1}{1 - y \tilde{m}(z)}} dH(\lambda) \quad (4.20)$$

Because the imaginary part of LHS of (4.20) has the same sign as  $z$ , we conclude that  $\Im(\tilde{m}(z))$  should have the same sign as  $\Im(z)$ .

Our next goal is to show that for every non-real  $z$ , the equation (4.20) has a unique solution  $\tilde{m}(z)$  whose imaginary part has the same sign as  $\Im(z)$ . By symmetry, we only need to consider the case where  $\Im(z) > 0$ . Suppose that there are two different solutions  $m_1(z) \neq m_2(z)$ . Making difference of both sides of (4.20), we obtain

$$\begin{aligned} 1 &= \int \frac{\frac{y}{(1 - y m_1)(1 - y m_2)}}{\left(\frac{\lambda}{z} - \frac{1}{1 - y m_1}\right)\left(\frac{\lambda}{z} - \frac{1}{1 - y m_2}\right)} dH(\lambda) \\ &\leq \left( \int \frac{\frac{y}{|1 - y m_1|^2}}{\left|\frac{\lambda}{z} - \frac{1}{1 - y m_1}\right|^2} dH(\lambda) \int \frac{\frac{y}{|1 - y m_2|^2}}{\left|\frac{\lambda}{z} - \frac{1}{1 - y m_2}\right|^2} dH(\lambda) \right)^{1/2}. \end{aligned} \quad (4.21)$$

Comparing the imaginary parts of both sides of (4.21), we have

$$\Im(m_j) = \int \frac{\frac{\Im(z)\lambda}{|z|^2} + \frac{y \Im(m_j)}{|1 - y m_j|^2}}{\left|\frac{\lambda}{z} - \frac{1}{1 - y m_j}\right|^2} dH(\lambda), \quad j = 1, 2.$$

Since  $\Im(m_j) > 0$  implies that

$$\int \frac{\frac{y}{|1-ym_j|^2}}{\left|\frac{\lambda}{z} - \frac{1}{1-ym_j}\right|^2} dH(\lambda) < 1,$$

which contradicts to (4.21).

The proof of the lemma is completed. ■

**Lemma 4.4** *Under Assumptions 1, 2, 3, we have*

$$s(z) = -\frac{1}{z} - \frac{1}{z^2} \cdot \tilde{s}(z)$$

where

$$\tilde{\underline{s}}(z) = \frac{-z}{1 - y\tilde{m}(z)} = \frac{-z}{1 - y \int \frac{1}{t/z - (1-y\tilde{m}(z))^{-1}} dH(t)} = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t+\tilde{\underline{s}}(z)} dH(t)},$$

$\tilde{\underline{s}}(z) = -(1-y)z + y\tilde{s}(z)$ ,  $s(z)$  is the Stieltjes transform of the LSD of  $\mathbf{S}^{-1}\mathbf{T}$ , and  $\tilde{s}(z)$  is the limit of  $\frac{1}{p}\text{tr}(\mathbf{S} - \frac{\mathbf{T}}{z})^{-1}\mathbf{T}$ .

PROOF. We have

$$\frac{1}{p}\text{tr}(\mathbf{S}^{-1}\mathbf{T} - z\mathbf{I})^{-1} = -\frac{1}{z} - \frac{1}{z^2} \cdot \frac{1}{p}\text{tr}\left(\mathbf{S} - \frac{\mathbf{T}}{z}\right)^{-1}\mathbf{T}$$

and

$$s_n(z) = \frac{1}{pz} \sum_{i=1}^n \frac{\boldsymbol{\alpha}'_i(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}\boldsymbol{\alpha}_i}{1 - \boldsymbol{\alpha}'_i(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}\boldsymbol{\alpha}_i} = -\frac{1}{yz} + \frac{1}{yz} \cdot \frac{1}{n} \sum_{i=1}^n \frac{1}{1 - \boldsymbol{\alpha}'_i(\frac{1}{z}\mathbf{T} - \mathbf{S}_i)^{-1}\boldsymbol{\alpha}_i}$$

where  $\boldsymbol{\alpha}_i = \frac{1}{\sqrt{n}}\mathbf{X}_i$ ,  $i = 1, \dots, n$ . Let the limit of  $\frac{1}{p}\text{tr}(\mathbf{S} - \frac{\mathbf{T}}{z})^{-1}\mathbf{T}$  be  $\tilde{s}(z)$  and  $\tilde{s}_n(z) = \frac{1}{p}\text{tr}(\mathbf{S} - \frac{\mathbf{T}}{z})^{-1}\mathbf{T}$ . Let

$$\tilde{\underline{s}}(z) = -(1-y)z + y\tilde{s}(z).$$

In fact, we have

$$s_n(z) = -\frac{1}{z} - \frac{1}{z^2} \tilde{s}_n(z) \tag{4.22}$$

$$s(z) = -\frac{1}{z} - \frac{1}{z^2} \cdot \tilde{s}(z) = -\frac{1}{yz} + \frac{1}{yz} \cdot \frac{1}{1 - y\tilde{m}(z)}$$

$$\tilde{\underline{s}}(z) = -(1-y)z + y\tilde{s}(z) = \frac{-z}{1 - y\tilde{m}(z)}, \quad E\tilde{\underline{s}}_n(z) = -zE\beta_i(z) \tag{4.23}$$

where  $\beta_i(z) = \frac{1}{1 - \alpha'_i(\frac{\mathbf{T}}{z} - \mathbf{S}_i)^{-1}\alpha_i}$ . Therefore, we have

$$\underline{\tilde{s}}(z) = \frac{-z}{1 - y\tilde{m}(z)} = \frac{-z}{1 - y \int \frac{1}{\frac{t}{z} - \frac{1}{1 - y\tilde{m}(z)}} dH(t)} = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \underline{\tilde{s}}(z)} dH(t)}.$$

That is,

$$\underline{\tilde{s}}(z) = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \underline{\tilde{s}}(z)} dH(t)}.$$

The proof of the lemma is completed. ■

Here we give some notes:

$$\tilde{m}(z) = \int \frac{1}{\frac{\lambda}{z} - \frac{1}{1 - y\tilde{m}(z)}} dH(\lambda) \quad (4.24)$$

where  $\tilde{m}(z)$  is the limit of  $\frac{1}{p}\text{tr}(\frac{1}{z}\mathbf{T} - \mathbf{S})^{-1}$  and  $H(t)$  is the LSD of  $\mathbf{T}$ .

$$\underline{\tilde{s}}_n^0(z) \rightarrow \underline{\tilde{s}}(z) = \frac{-z}{1 - y\tilde{m}(z)}, \quad 1 + yzs(z) = \frac{1}{1 - y\tilde{m}(z)} = \frac{1}{1 - y \int \frac{dH(t)}{t/z - (1 + yzs(z))}} \quad (4.25)$$

where  $s(z)$  is the Stieltjes transform of the LSD of  $\mathbf{S}^{-1}\mathbf{T}$  and  $\underline{\tilde{s}}_n^0(z) = \frac{1}{-\frac{1}{z} + y \int \frac{1}{t + \underline{\tilde{s}}_n^0(z)} dH_n(t)}$  with the ESD  $H_n(t)$  of  $\mathbf{T}$ .

$$E\beta_1(z) \rightarrow \frac{1}{1 - y\tilde{m}(z)} = 1 + yzs(z), \quad b_i(z) \rightarrow \frac{1}{1 - y\tilde{m}(z)} = 1 + yzs(z) \quad (4.26)$$

where  $\beta_1(z) = \frac{1}{1 - \alpha'_1(\frac{\mathbf{T}}{z} - \mathbf{S}_1)^{-1}\alpha_1}$  and  $b_1(z) = \frac{1}{1 - n^{-1}\text{Etr}(\frac{1}{z}\mathbf{T} - \mathbf{S}_1)^{-1}}$ .

$$-z(1 + yzs(z)) = \frac{-z}{1 - y \int \frac{dH(t)}{t/z - (1 + yzs(z))}} = \frac{-z}{1 - y \int \frac{zdH(t)}{t - z(1 + yzs(z))}}. \quad (4.27)$$

$$-\frac{1}{z} + y \int \frac{dH(t)}{t - z(1 + yzs(z))} = \frac{1}{-z(1 + yzs(z))} \quad (4.28)$$

$$\frac{1}{z^2} - y \int \frac{(-z(1 + yzs(z)))' dH(t)}{(t - z(1 + yzs(z)))^2} = \frac{-(-z(1 + yzs(z)))'}{(-z(1 + yzs(z)))^2} \quad (4.29)$$

$$\frac{(z(1 + yzs(z)))^2}{z^2} - y \int \frac{(z(1 + yzs(z)))^2 dH(t)}{(t - z(1 + yzs(z)))^2} (-z(1 + yzs(z)))' = -(-z(1 + yzs(z)))' \quad (4.30)$$

$$(-z(1 + yzs(z)))' = \frac{-1}{z^2} \cdot \frac{(-z(1 + yzs(z)))^2}{1 - y \int \frac{(-z(1 + yzs(z)))^2 dH(t)}{(t - z(1 + yzs(z)))^2}}. \quad (4.31)$$

Especially, when  $\mathbf{T} = \mathbf{I}_p$ , by (2.9), (4.25), (4.20) and the definition of  $\tilde{m}(z)$ , we have

$$1 + yzs(z) = \frac{1}{1 + y \cdot m(\frac{1}{z})} = -\frac{1}{z} \underline{m}(\frac{1}{z}) = (1 - y) - \frac{y}{z} \cdot m(\frac{1}{z})$$

and

$$\tilde{m}(z) = -m(\frac{1}{z}) = \frac{1}{\frac{1}{z} - \frac{1}{1 + y \cdot m(\frac{1}{z})}}.$$

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